MMME2046 Dynamics: Control Lecture 4

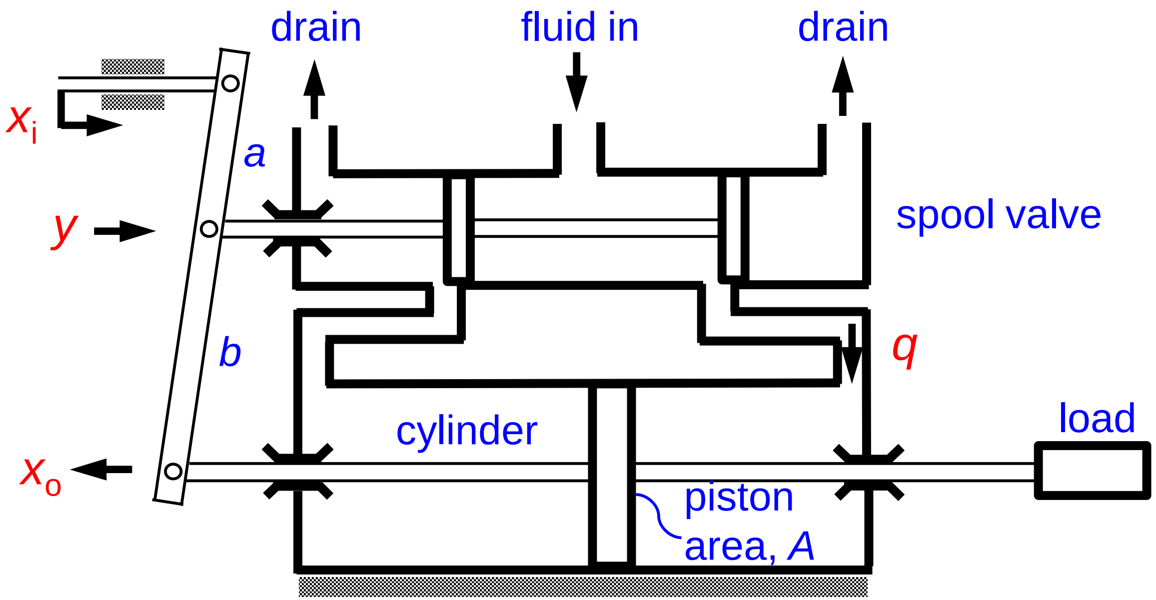
Position Control Systems

(case studies in 1st & 2nd order systems)

Lecture Objectives:

- Introduce the differences between 1st and 2nd order systems
- Analyse steady-state responses under step and ramp inputs
- Analyse transient behaviour through the roots of the characteristic equations

Recap: Hydraulic Position Control System

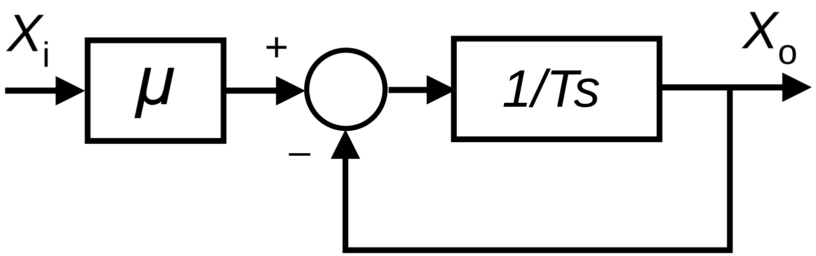


It was shown that the **transfer function** is given by

$$G(s) = \frac{X_o(s)}{X_i(s)} = \frac{\mu}{1 + Ts}$$

1st order system

with the **block diagram**



Hydraulic Position Control System: Equations for the Model

Spool Valve

in the time domain $$q = Ky$$

transfer function $$\frac{Q(s)}{Y(s)} = K \qquad (1)$$

Ram Piston

in the time domain $$A\frac{dx_o}{dt} = q$$

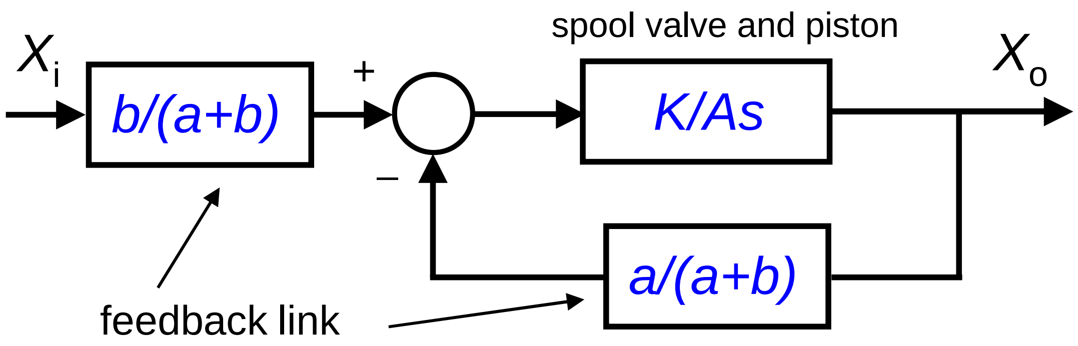
transfer function $$\frac{X_o(s)}{Q(s)} = \frac{1}{As} \qquad (2)$$

Feedback Link

in the time domain $$y = \frac{b}{a+b}x_i - \frac{a}{a+b}x_o$$

transfer function $$Y(s) = \frac{b}{a+b}X_i(s) - \frac{a}{a+b}X_o(s) \qquad (3)$$

Hydraulic Position Control System: Overall Transfer Function



From the block diagram

$$X_o(s) = \left[X_i(s)\frac{b}{a+b} - X_o(s)\frac{a}{a+b}\right]\frac{K}{As}$$

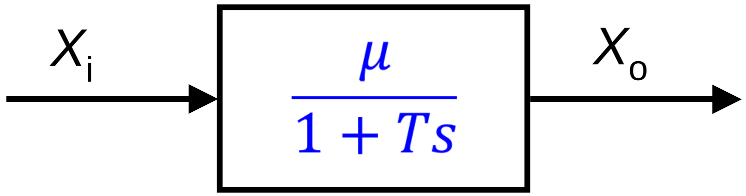
rearranging

$$\left[1 + \frac{A(a+b)s}{Ka}\right]X_o(s) = \frac{b}{a}X_i(s)$$

$$\frac{X_o(s)}{X_i(s)} = \frac{\mu}{1+Ts} \qquad (4)$$

First order system with time constant T and gain μ

Hydraulic Position Control System: Control System Model

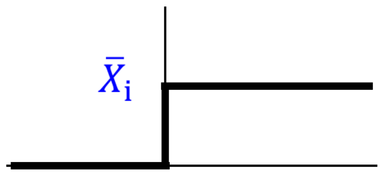


$$T = \frac{A(a+b)}{Ka}$$ **time constant**

$$\mu = \frac{b}{a}$$ **steady-state gain**

Hydraulic Position Control System under Standard Inputs

i) step Input



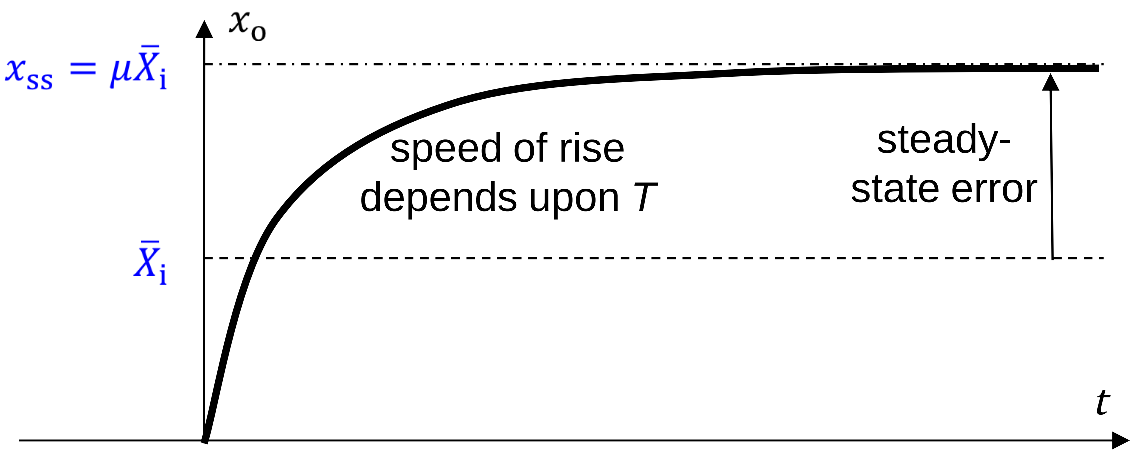
$t < 0 \qquad x_i(t) = 0$

$t \geq 0 \qquad x_i(t) = \bar{X}_i$

From the table of L.T. $$X_i(s) = \frac{\bar{X}_i}{s} \qquad (5)$$

The output in s-domain $$X_o(s) = \frac{\mu \bar{X}_i}{s(1+Ts)} \qquad (6)$$

In the time domain $$x_o(t) = \mu \bar{X}_i \left(1 - e^{-\frac{t}{T}}\right) \qquad (7)$$



Hydraulic Position Control System under Standard Inputs

ii) Ramp Input

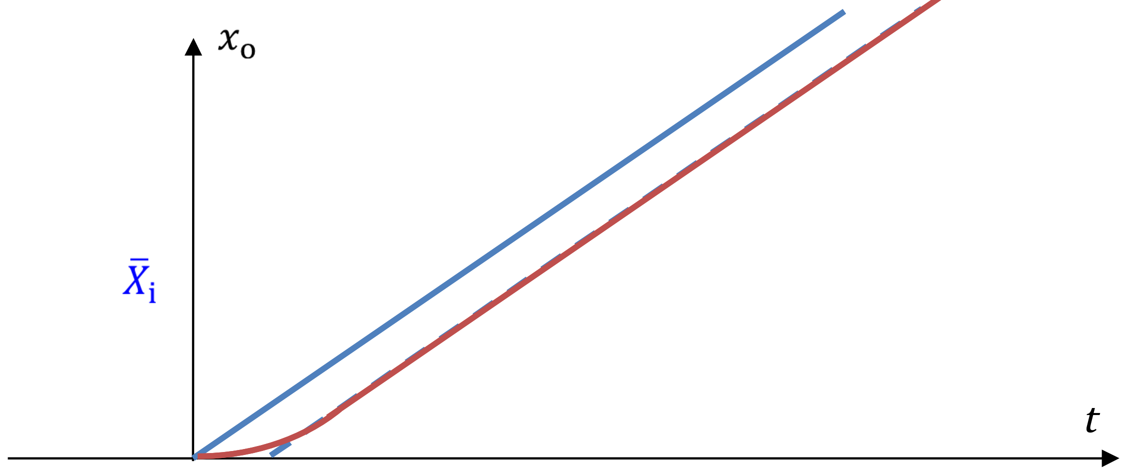
$t < 0 \qquad x_i(t) = 0$

$t \geq 0 \qquad x_i(t) = \bar{V}_i t$

From the the table of L.T. $\quad X_i(s) = \dfrac{\bar{V}_i}{s^2}$

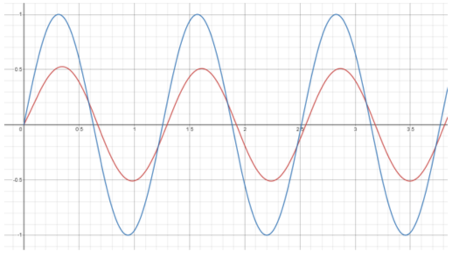
The output in s-domain $\quad X_o(s) = \dfrac{\mu \bar{V}_i}{s^2(1 + Ts)}$

In the time domain $\quad x_o(t) = \mu \bar{V}_i t - \mu \bar{V}_i T \left(1 - e^{-t/T}\right)$



Hydraulic Position Control System under Standard Inputs

iii) Oscillatory input



$$t < 0 \qquad x_i(t) = 0$$
$$t \geq 0 \qquad x_i(t) = A\sin(\omega t)$$

From the table of L.T.

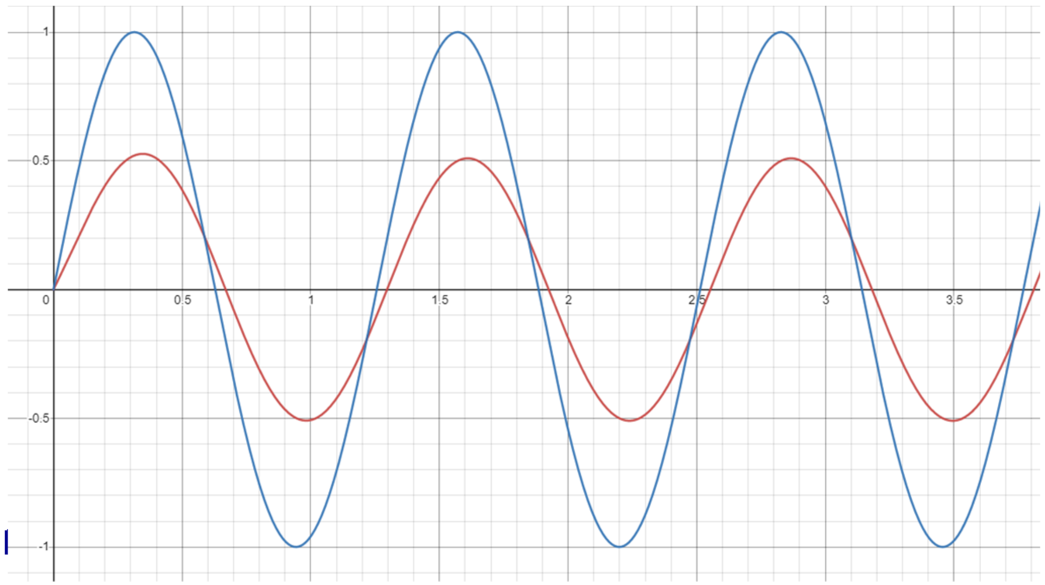
$$X_i(s) = \frac{A\omega^2}{(s^2+\omega^2)} \tag{5}$$

The output in s-domain

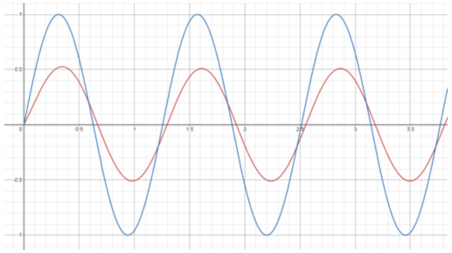
$$X_o(s) = \frac{\mu A\omega^2}{(s^2+\omega^2)(1+Ts)} \tag{6}$$

$$X_{out}(s) = \frac{\mu A\omega^2}{1+\omega^2T^2}\left(\frac{1-Ts}{(s^2+\omega^2)} + \frac{T^2}{(1+Ts)}\right) = \frac{\mu A\omega^2}{1+\omega^2T^2}\left(\frac{1}{(s^2+\omega^2)} - \frac{Ts}{(s^2+\omega^2)} + \frac{T^2}{(1+Ts)}\right)$$

Steady state: gain and phase angle (1st order lag)



iii) Oscillatory input



$t < 0 \qquad x_i(t) = 0$

$t \geq 0 \qquad x_i(t) = A\sin(\omega t)$

From the table of L.T.

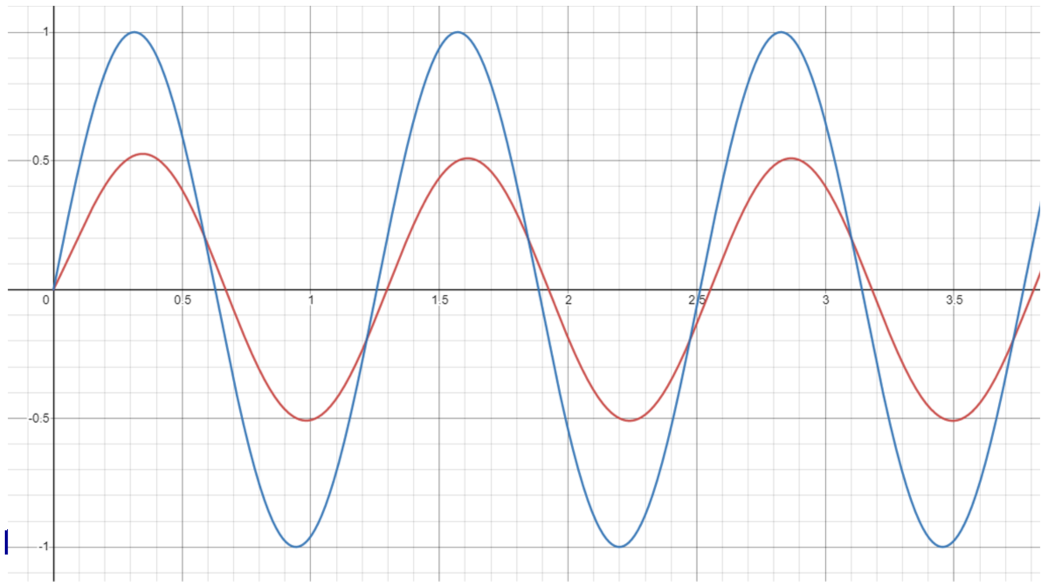
$$X_i(s) = \frac{A\omega^2}{(s^2 + \omega^2)} \tag{5}$$

The output in s-domain

$$X_o(s) = \frac{\mu A\omega^2}{(s^2 + \omega^2)(1 + Ts)} \tag{6}$$

$$X_{out}(s) = \frac{\mu A\omega^2}{1+\omega^2T^2}\left(\frac{1-Ts}{(s^2+\omega^2)} + \frac{T^2}{(1+Ts)}\right) = \frac{\mu A\omega^2}{1+\omega^2T^2}\left(\frac{1}{(s^2+\omega^2)} - \frac{Ts}{(s^2+\omega^2)} + \frac{T^2}{(1+Ts)}\right)$$

Steady state: gain and phase angle (1st order lag)

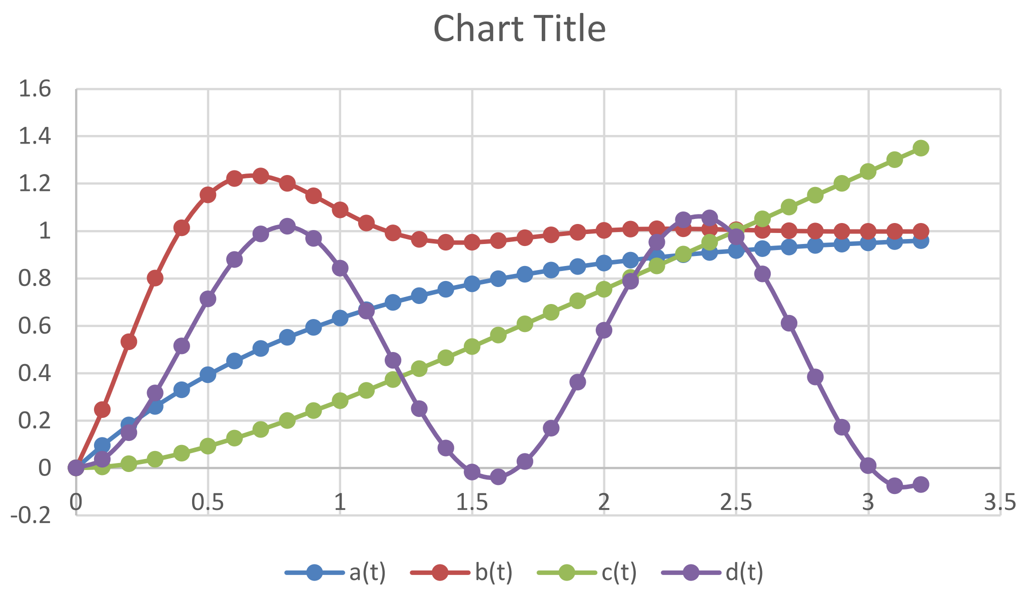


Recap: The Final Value Theorem

The **final value theorem:**

$$x_{ss} = \lim_{t \to \infty} x_o(t) = \lim_{s \to 0} sX_o(s) \quad (9)$$

Gives the steady-state response of a system.
Some provisos:
Steady state implies that we have a finite end value:



Which of these can we use the finite value theorem on?
a(t)?
b(t)?
c(t)?
d(t)?

Hydraulic Position Control System: The Final Value Theorem

The **final value theorem** gives the **steady-state response**

$$x_{ss} = \lim_{t \to \infty} x_o(t) = \lim_{s \to 0} sX_o(s) \qquad (9)$$

$$x_{ss} = \lim_{s \to 0} s \frac{\mu \bar{X}_i}{s(1+Ts)} = \mu \bar{X}_i \qquad (10)$$

The **steady-state error** from the **final value theorem**

$$E(s) = X_i(s) - X_o(s)$$

$$e_{ss} = \lim_{s \to 0} s \frac{(1+Ts-\mu)}{(1+Ts)} \frac{\bar{X}_i}{s} = \lim_{s \to 0} \frac{(1+Ts-\mu)}{(1+Ts)} \bar{X}_i = (1-\mu)\bar{X}_i \qquad (13)$$

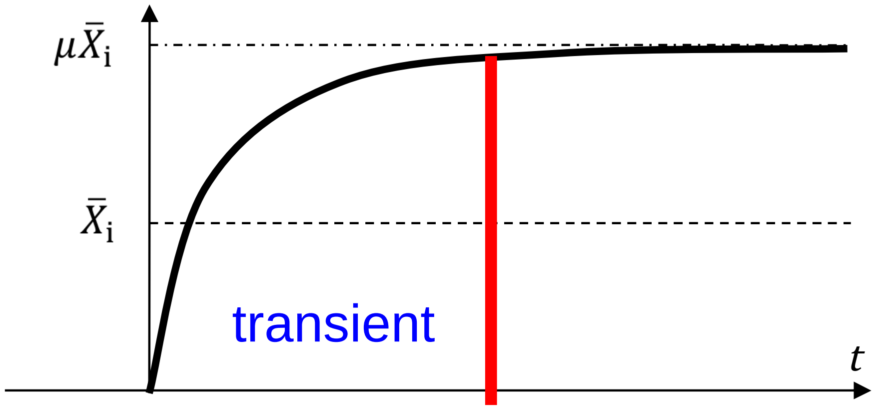
Exercise: Show that if the error is defined as

$$e_{new} = \mu x_i - x_o$$

the steady-state error is zero

$$e_{new\_ss} = 0$$

Hydraulic Position Control System: Transient Response



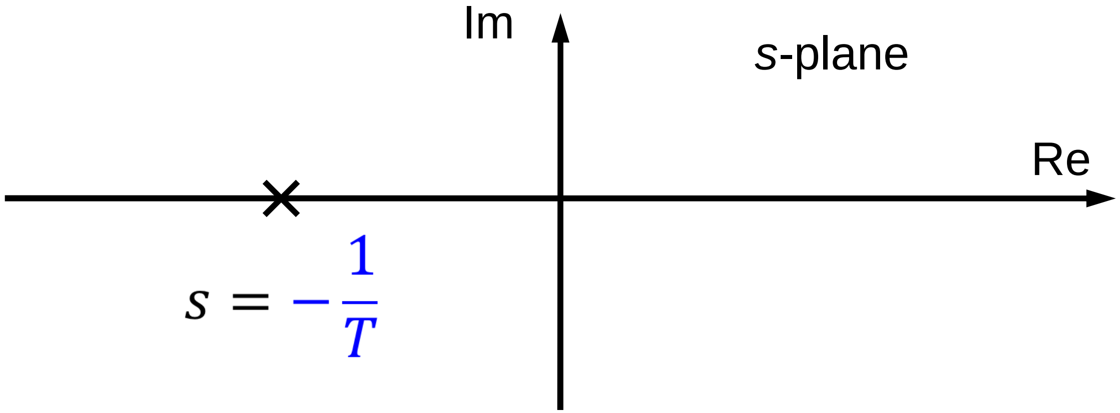
characteristic equation

Set denominator =0

$$P(s) = s + \frac{1}{T} = 0$$

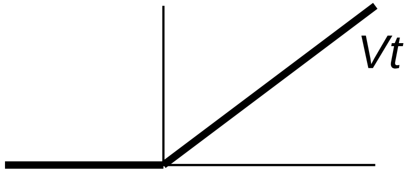
with one real root $s = -\frac{1}{T}$

The **roots of the C.E.** in the **s-plane** govern stability and transient behaviour



Hydraulic Position Control System under Standard Inputs

ii) ramp Input



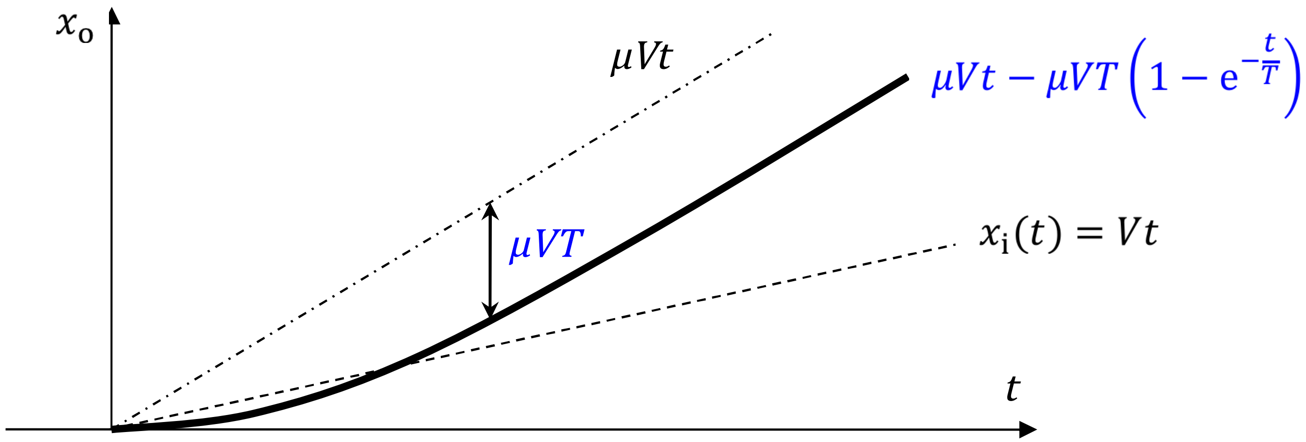
$t < 0 \qquad x_\mathrm{i}(t) = 0$

$t \geq 0 \qquad x_\mathrm{i}(t) = Vt$

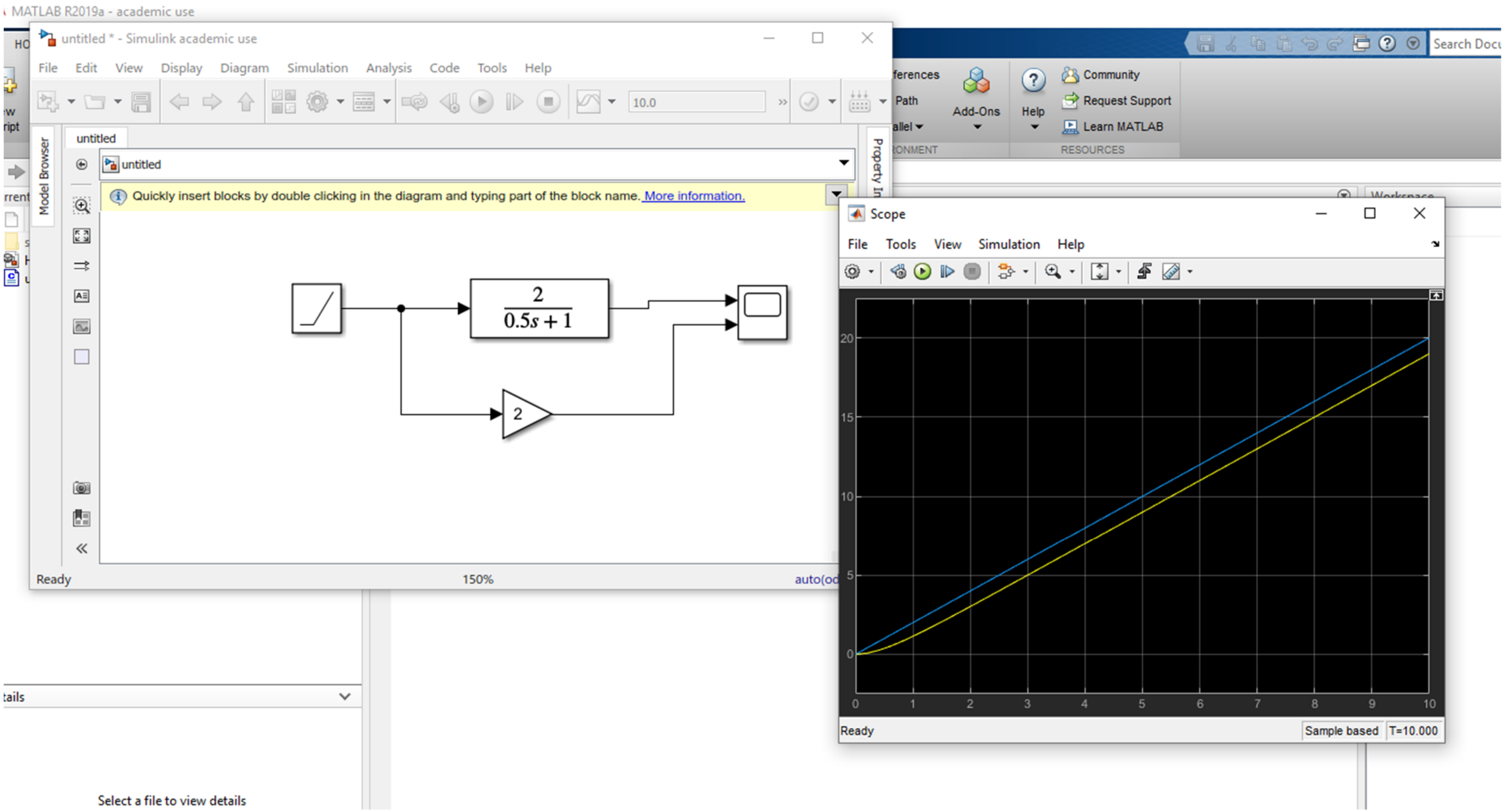
From the table of L.T. $$X_\mathrm{i}(s) = \frac{V}{s^2} \qquad (14)$$

The output in s-domain $$X_\mathrm{o}(s) = \frac{\mu V}{s^2(1+Ts)} \qquad (15)$$

In the time domain $$x_\mathrm{o}(t) = \mu Vt - \mu VT\left(1 - \mathrm{e}^{-\frac{t}{T}}\right) \qquad (16)$$

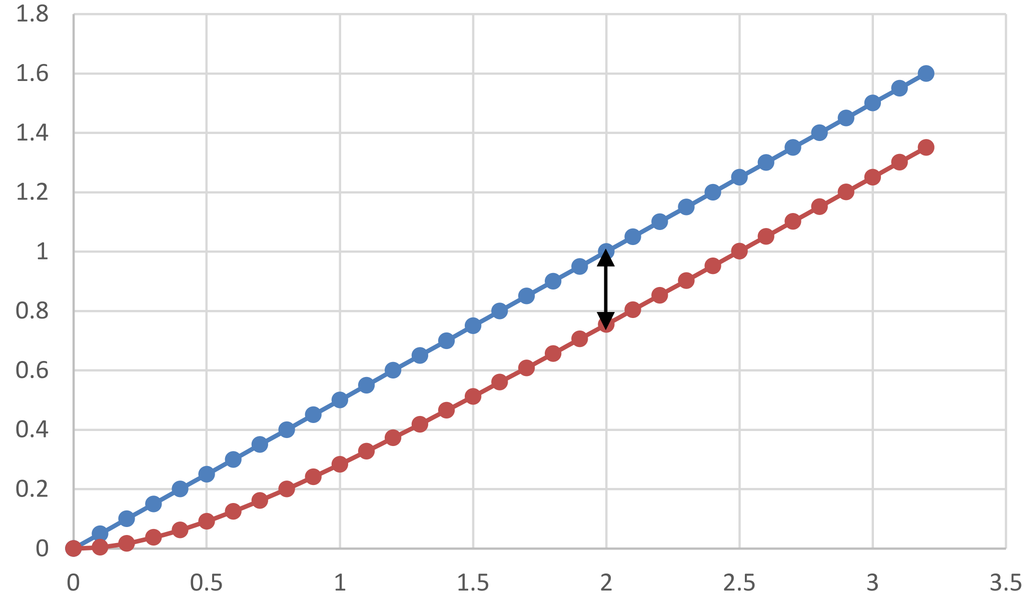


Simulink model

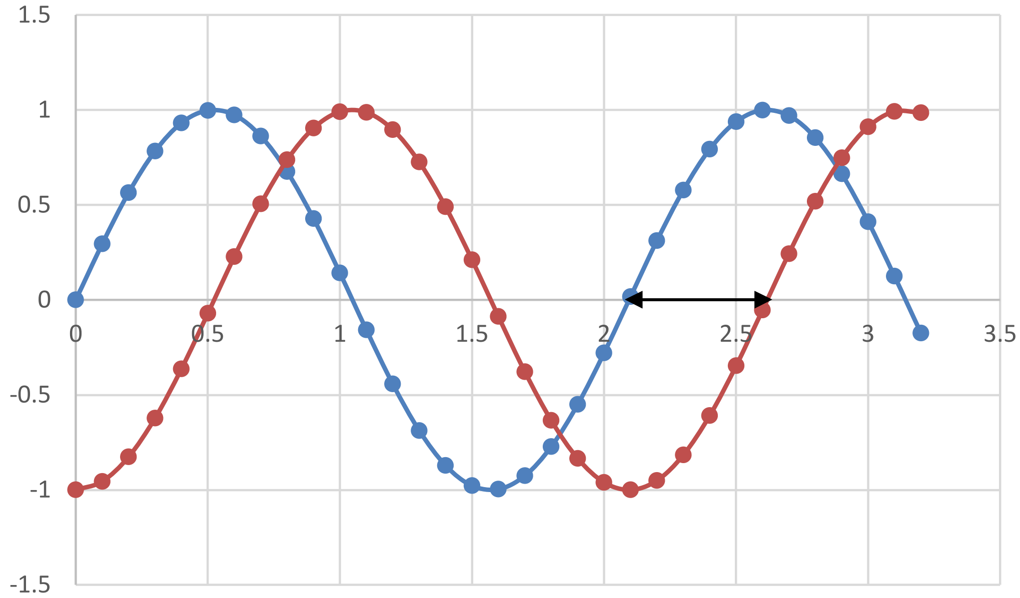


Other types of Steady State Error

Ramp output: Velocity lag



Oscillating output: phase lag
(not in scope!)



Hydraulic Position Control System: S.-S. Error under Ramp Input

After a "large" time interval (t > 4T) $\quad$ $\mathrm{e}^{-t/T} \to 0$

$$x_{ss}(t) = \mu V(t - T)$$

Define the error as

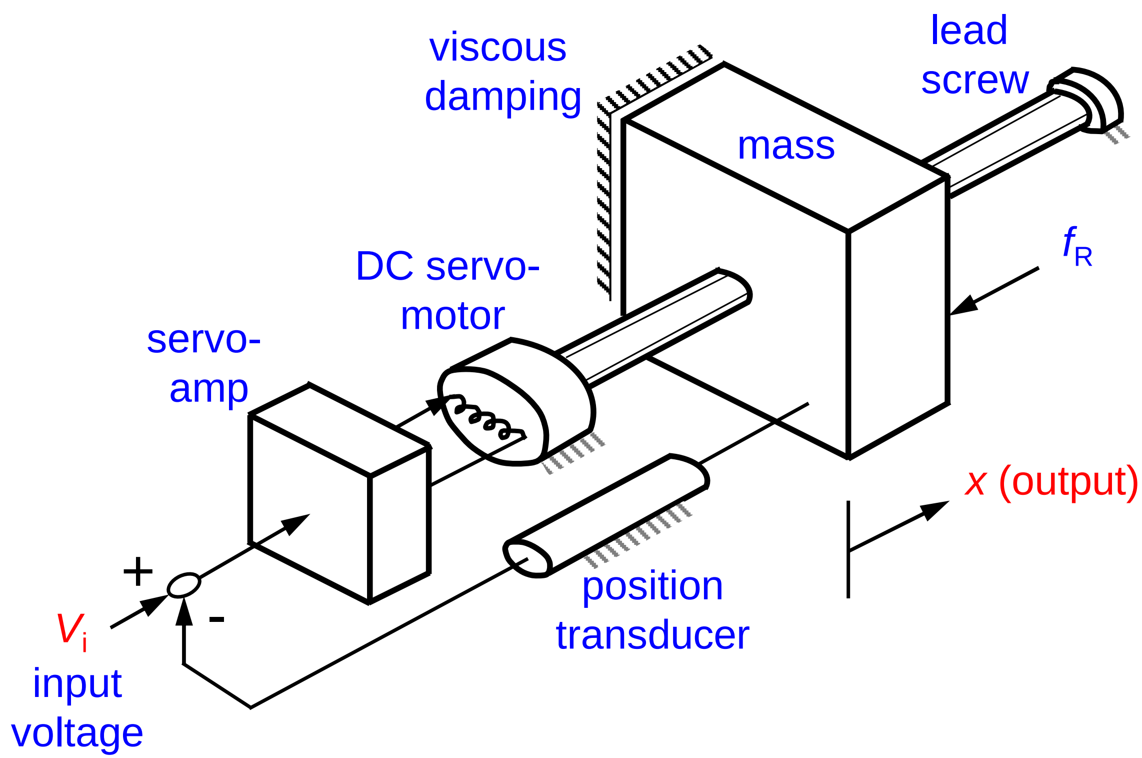
$$E(s) = \mu X_i(s) - X_o(s)$$

$$e_{ss} = \lim_{s\to 0} s\left(\mu - \frac{\mu}{1+Ts}\right) \; X_i(s) = \lim_{s\to 0} s\left(\frac{\mu s T}{1+Ts}\right) \; X_i(s)$$

$$e_{ss} = \lim_{s\to 0} s\frac{\mu s T}{(1+Ts)}\frac{V}{s^2} = \mu V T$$

non-zero **finite** steady-state error called "velocity lag"

Example: Electro-Mechanical Position Control System



It will be shown that the **transfer functions** may be written as

$$\frac{X(s)}{X_i(s)} = \frac{\omega_n^2}{s^2 + 2\gamma\omega_n s + \omega_n^2}$$

$$\frac{X(s)}{F_R(s)} = \frac{-1}{M(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

2nd order system

https://www.youtube.com/watch?v=Sn8DqDGwazs

E.-M. Position Control System: Equations for the Model

i) **Position Transducer** output $V_x = K_4 x$ K_4 is constant

error voltage $V_e = V_i - V_x = V_i - K_4 x$

ii) **Servo-Amplifier** develops current (K_1 is another constant)

$$i_f = K_1 V_e = K_1 (V_i - K_4 x)$$

iii) **DC Servo-Motor** develops torque (K_2 is motor constant)

$$l_m = K_2 i_f = K_2 K_1 (V_i - K_4 x)$$

iv) At **Lead Screw** the torque is converted into a force on the load mass

$$f_m = K_3 l_m = K_3 K_2 K_1 (V_i - K_4 x) \qquad K_3 = 2\pi/(\text{pitch of leadscrew})$$

Laplace domain $$F_m(s) = K_1 K_2 K_3 (V_i(s) - K_4 X(s)) \qquad (1)$$

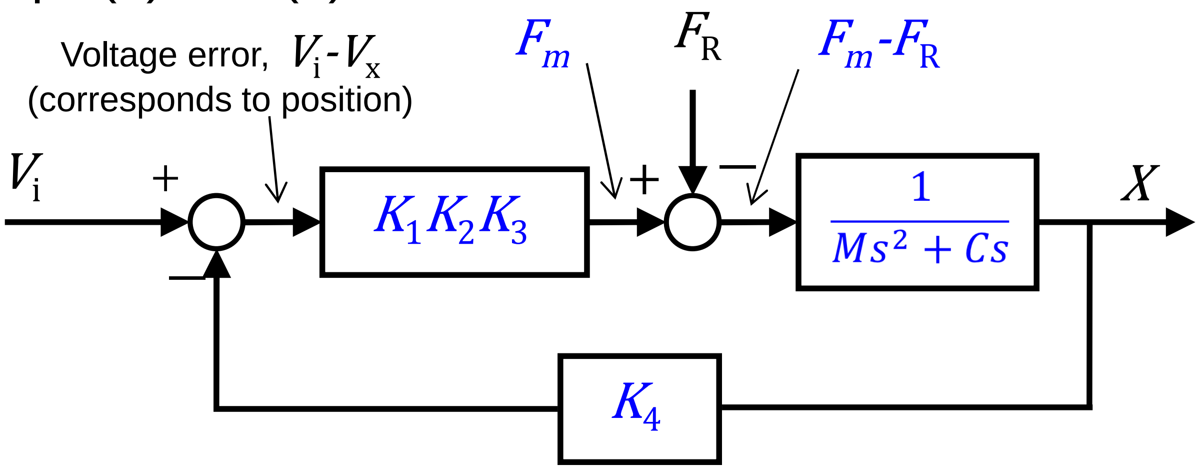
v) For the **Load Mass** assuming viscous damping

$$M\ddot{x} + C\dot{x} = f_m - f_\mathrm{R}$$

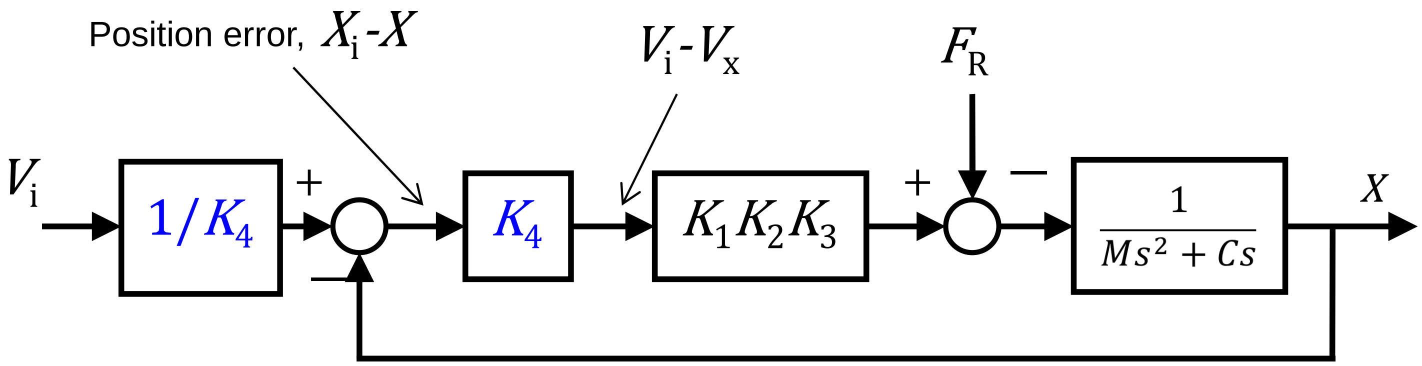
Laplace domain $$X(s) = \frac{F_m(s) - F_\mathrm{R}(s)}{Ms^2 + Cs} \qquad (2)$$

E.-M. Position Control System: Block Diagrams

Using Eqs. (1) and (2)



Focusing on actual position error



with $K = K_1K_2K_3K_4$

$$X(s) = ([X_i(s) - X(s)]K - F_R(s))\frac{1}{Ms^2 + Cs}$$

E-.M. Position Control System: Overall Transfer Function

Rearranging $$[Ms^2 + Cs + K]X(s) = KX_\mathrm{i}(s) - F_\mathrm{R}(s) \qquad (3)$$

Preferred form

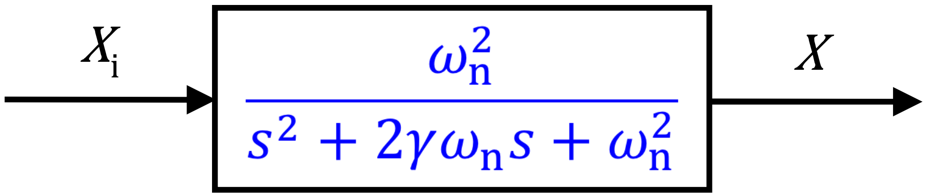
$$[s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2]X(s) = \omega_\mathrm{n}^2 X_\mathrm{i}(s) - \frac{F_\mathrm{R}(s)}{M}$$

with $$\frac{C}{M} = 2\gamma\omega_\mathrm{n} \quad \text{and} \quad \omega_\mathrm{n}^2 = \frac{K}{M}$$

$$X(s) = \frac{\omega_n^2 X_i(s)}{s^2 + 2\gamma\omega_n s + \omega_n^2} - \frac{F_R(s)}{M(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

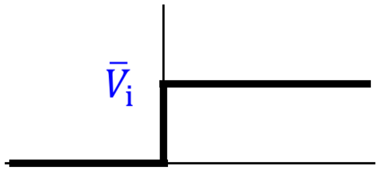
Transfer function

$$\frac{X(s)}{X_\mathrm{i}(s)} = \frac{\omega_\mathrm{n}^2}{s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2} \qquad (4)$$



E.-M. Position Control System under Standard Inputs

i) step Input



$t < 0 \qquad V_i(t) = 0$

$t \geq 0 \qquad V_i(t) = \bar{V}_i$

From the table of L.T.

$$X_i(s) = \frac{\bar{V}_i}{K_4 s} = \frac{\bar{X}_i}{s} \tag{5}$$

The output in s-domain

$$X_o(s) = \frac{\omega_n^2 \bar{X}_i}{s(s^2 + 2\gamma\omega_n s + \omega_n^2)} = \frac{\omega_n^2 \bar{X}_i}{s(s - p_1)(s - p_2)} \tag{6}$$

with the roots of the characteristic equation

$$s^2 + 2\gamma\omega_n s + \omega_n^2 = 0$$

$$p_1 = -\gamma\omega_n + \omega_n\sqrt{\gamma^2 - 1} \qquad p_2 = -\gamma\omega_n - \omega_n\sqrt{\gamma^2 - 1}$$

E.-M. Position Control System under Step Input

Assuming a *unit step input* and using partial fractions

$$X_o(s) = \frac{B}{s} + \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2}$$

where (for $\gamma \neq 1$)

$$B = 1\ ;\ \ A_1 = -\frac{1}{2} - \frac{\gamma}{2\sqrt{\gamma^2 - 1}}\ ;\ \ A_2 = -\frac{1}{2} + \frac{\gamma}{2\sqrt{\gamma^2 - 1}}$$

With the inverse Laplace transform, in the time domain

$$x_o(t) = B + A_1 e^{p_1 t} + A_2 e^{p_2 t} \qquad (7)$$

This solution, valid for $\gamma \neq 1$, gives rise to two distinct types of transient response.

E.-M. Position Control System under Step Input

i) $\gamma > 1$ p_1 and p_2 are **real** and **unequal**. For this situation the response is overdamped (non-oscillatory).

ii) $\gamma < 1$ p_1 and p_2 are **complex conjugate** (as A_1 and A_2)

$$p_1 = -\gamma\omega_\mathrm{n} + \mathrm{i}\omega_\mathrm{n}\sqrt{1-\gamma^2}$$

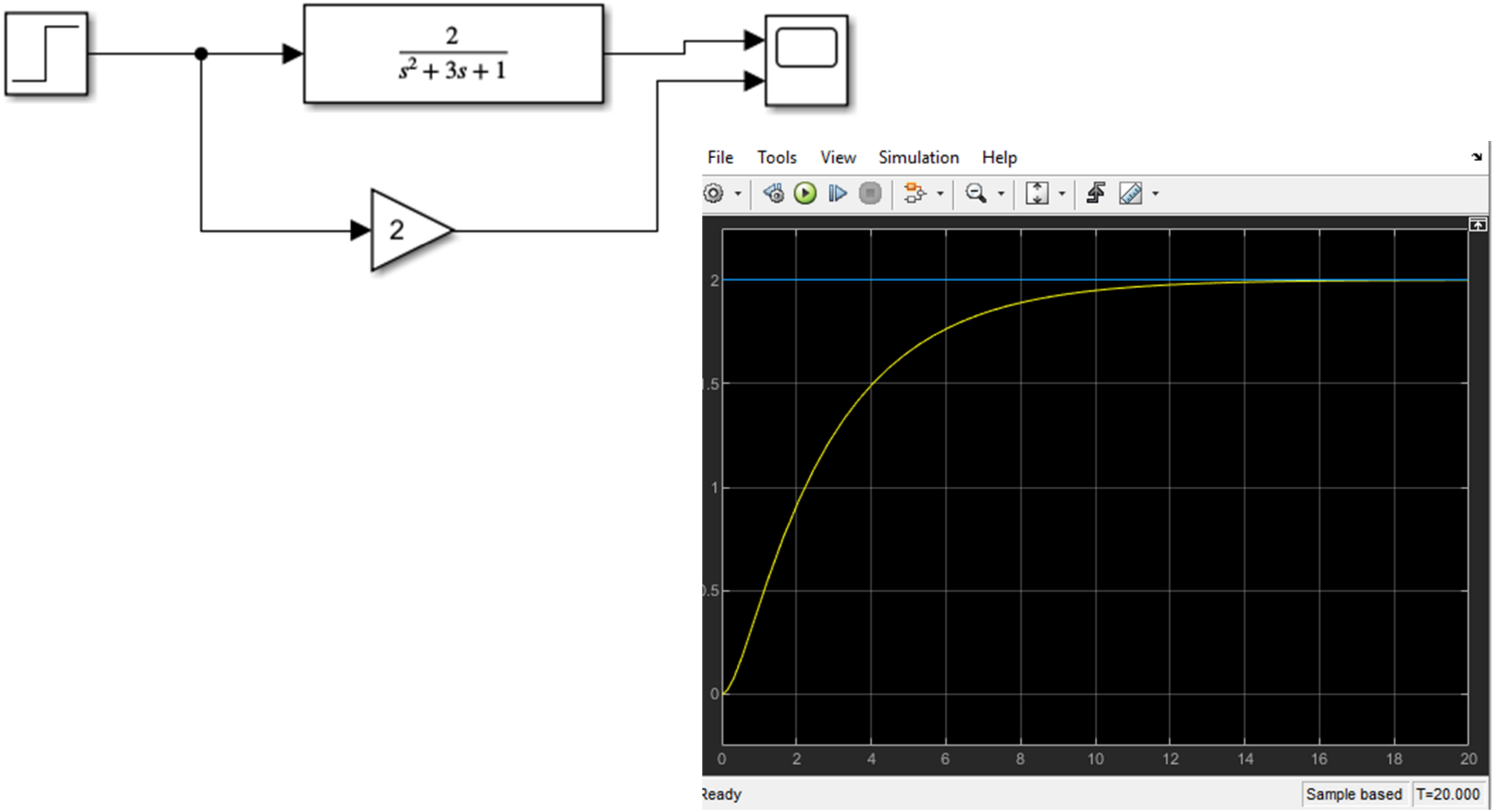
$$p_2 = -\gamma\omega_\mathrm{n} - \mathrm{i}\omega_\mathrm{n}\sqrt{1-\gamma^2}$$

$$x_\mathrm{o}(t) = \bar{X}_\mathrm{i}\left[1 - \frac{\mathrm{e}^{-\gamma\omega_\mathrm{n}t}}{\sqrt{1-\gamma^2}}\sin\left(\omega_\mathrm{n}t\sqrt{1-\gamma^2}+\phi\right)\right]$$

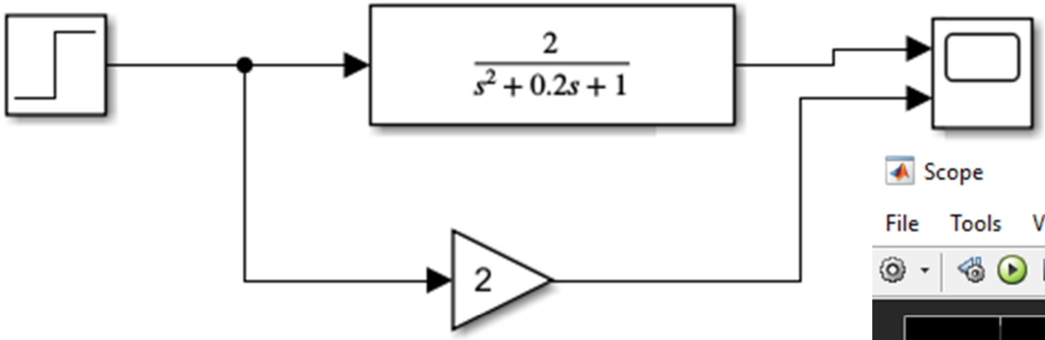
Maximum overshoot at $$t = \frac{\pi}{\omega_\mathrm{n}\sqrt{1-\gamma^2}}$$

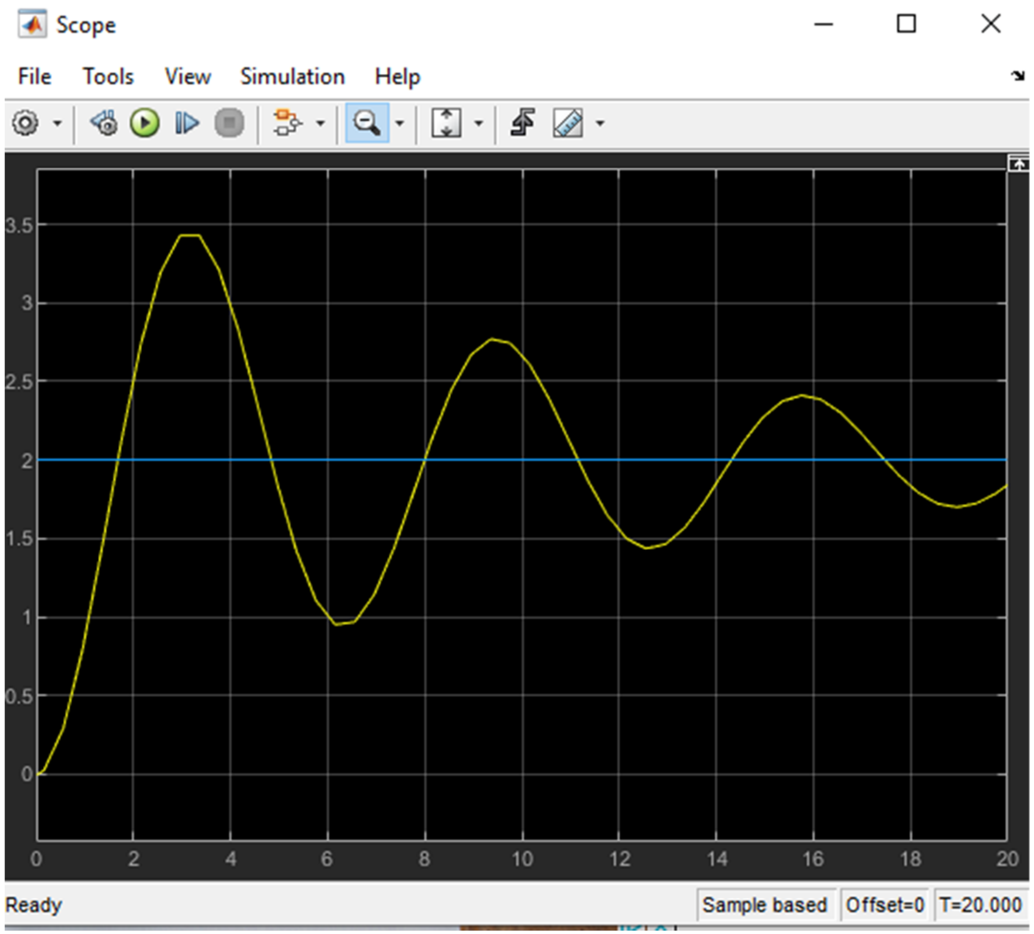
with magnitude $$x_\mathrm{max} = \bar{X}_\mathrm{i}\left(1 + \mathrm{e}^{\frac{-\gamma\pi}{\sqrt{1-\gamma^2}}}\right)$$

Simulink Model: γ>1



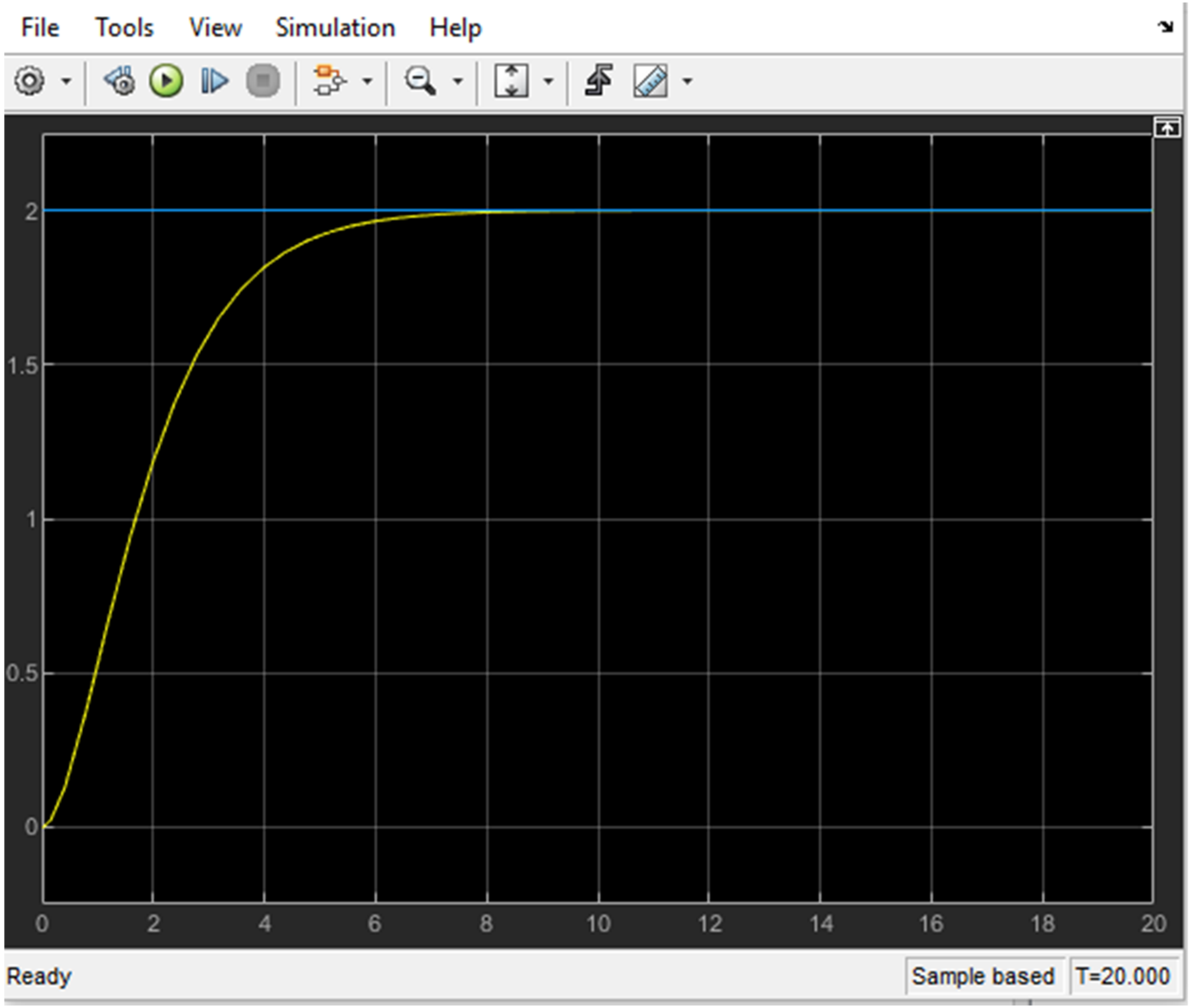
Simulink Model: γ<1





Simulink Model: γ=1

What is the transfer function in this case?

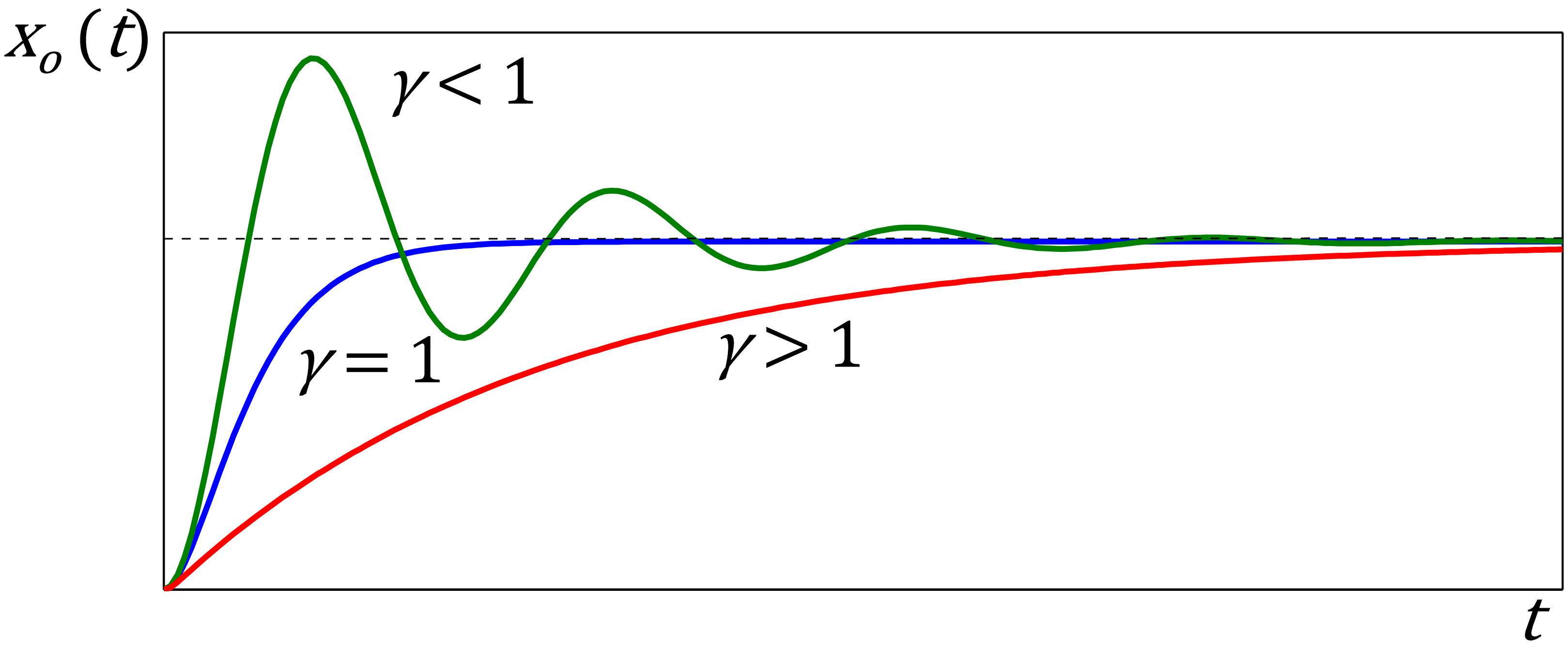


E.-M. Position Control System under Step Input

iii) $\gamma = 1$ p_1 and p_2 are **real** and **equal** (= - ω_n) and the response is Is said to be critically damped.

$$x_o(t) = \bar{X}_i[1 - (1 + \omega_n t)e^{-\omega_n t}]$$

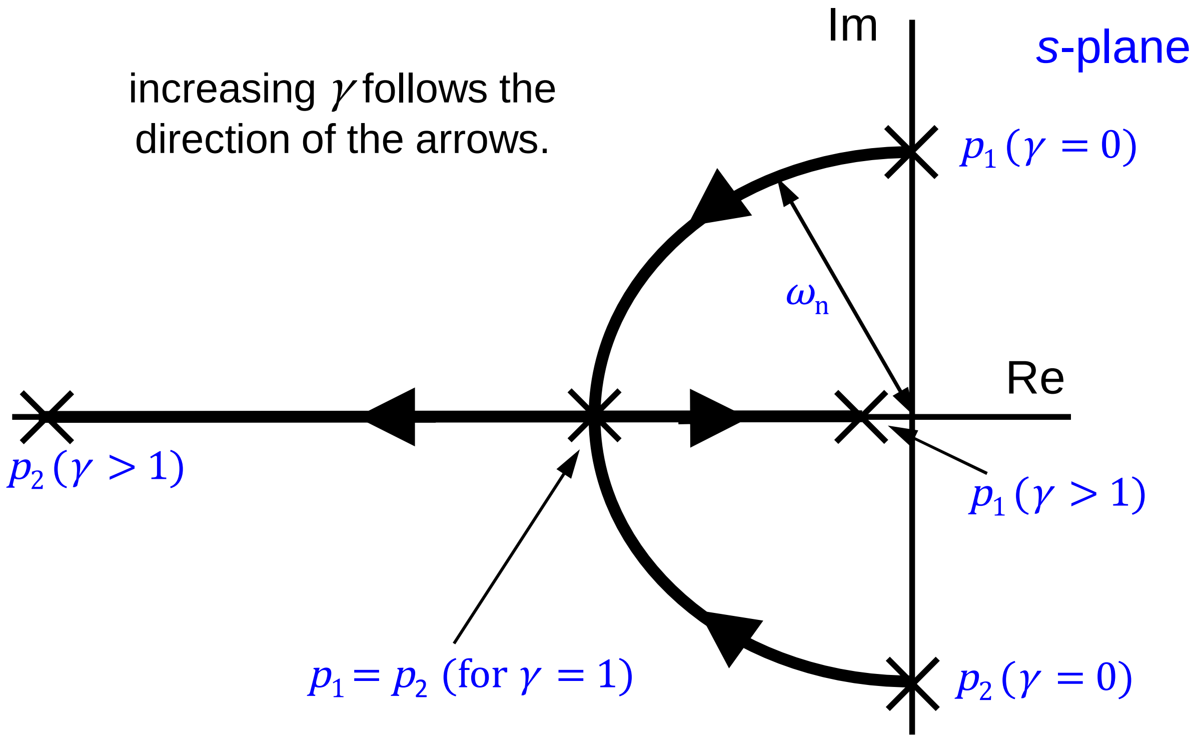
The transient responses under a step input for all three cases can be summarised



E.-M. Position Control System: Transient Response

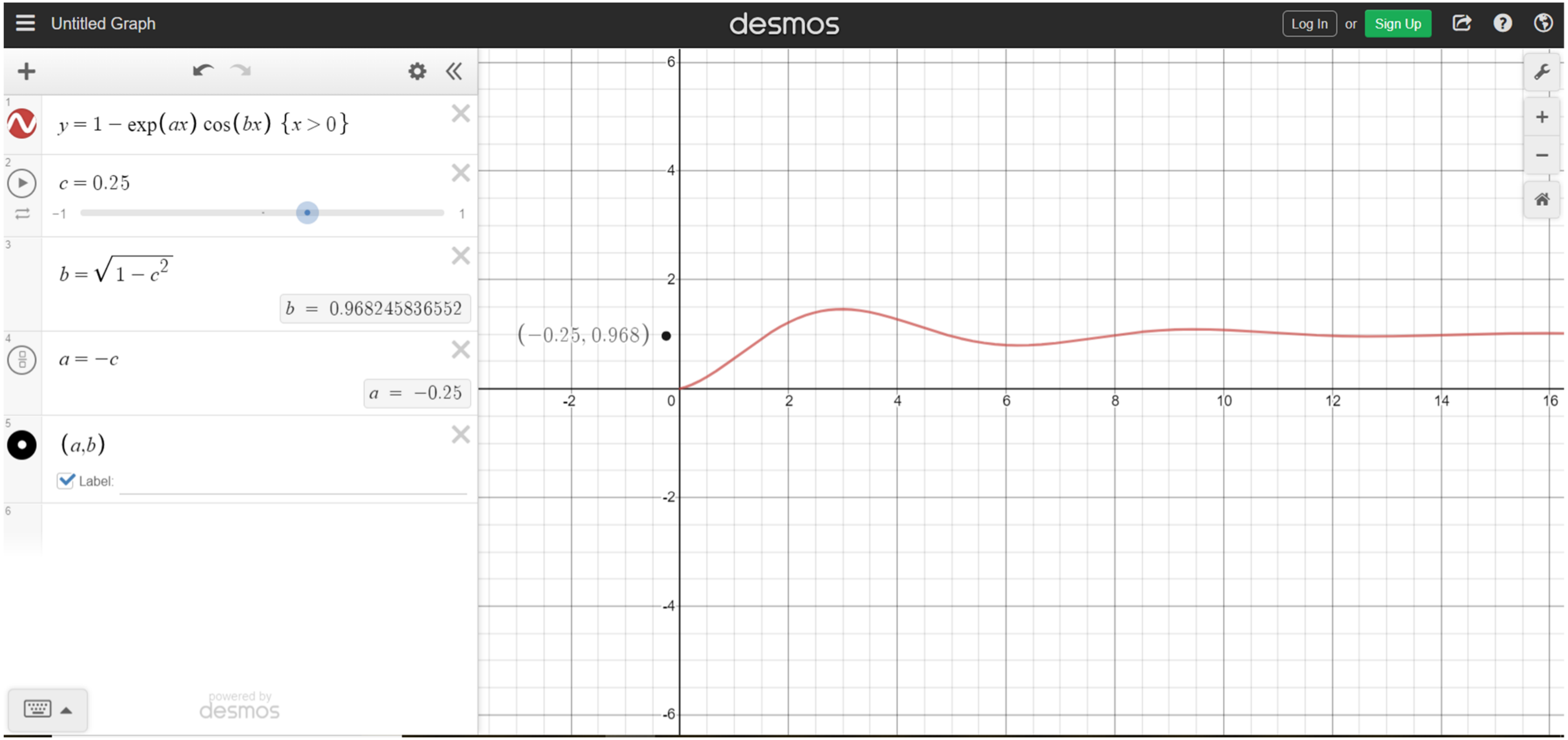
The **roots of the C.E.** in the **s-plane** govern stability and transient behaviour

$$p_1 = -\gamma\omega_n + \omega_n\sqrt{\gamma^2 - 1} \qquad p_2 = -\gamma\omega_n - \omega_n\sqrt{\gamma^2 - 1}$$



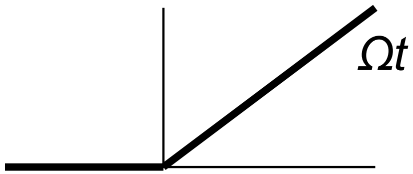
The roots trace out **loci** in the s-plane.

Visualisation of root locus



E.-M. Position Control System under Standard Inputs

ii) ramp Input



$t < 0 \qquad V_\mathrm{i}(t) = 0$

$t \geq 0 \qquad V_\mathrm{i}(t) = \Omega t$

From the table of L.T. $\qquad V_\mathrm{i}(s) = \frac{\Omega}{s^2}$

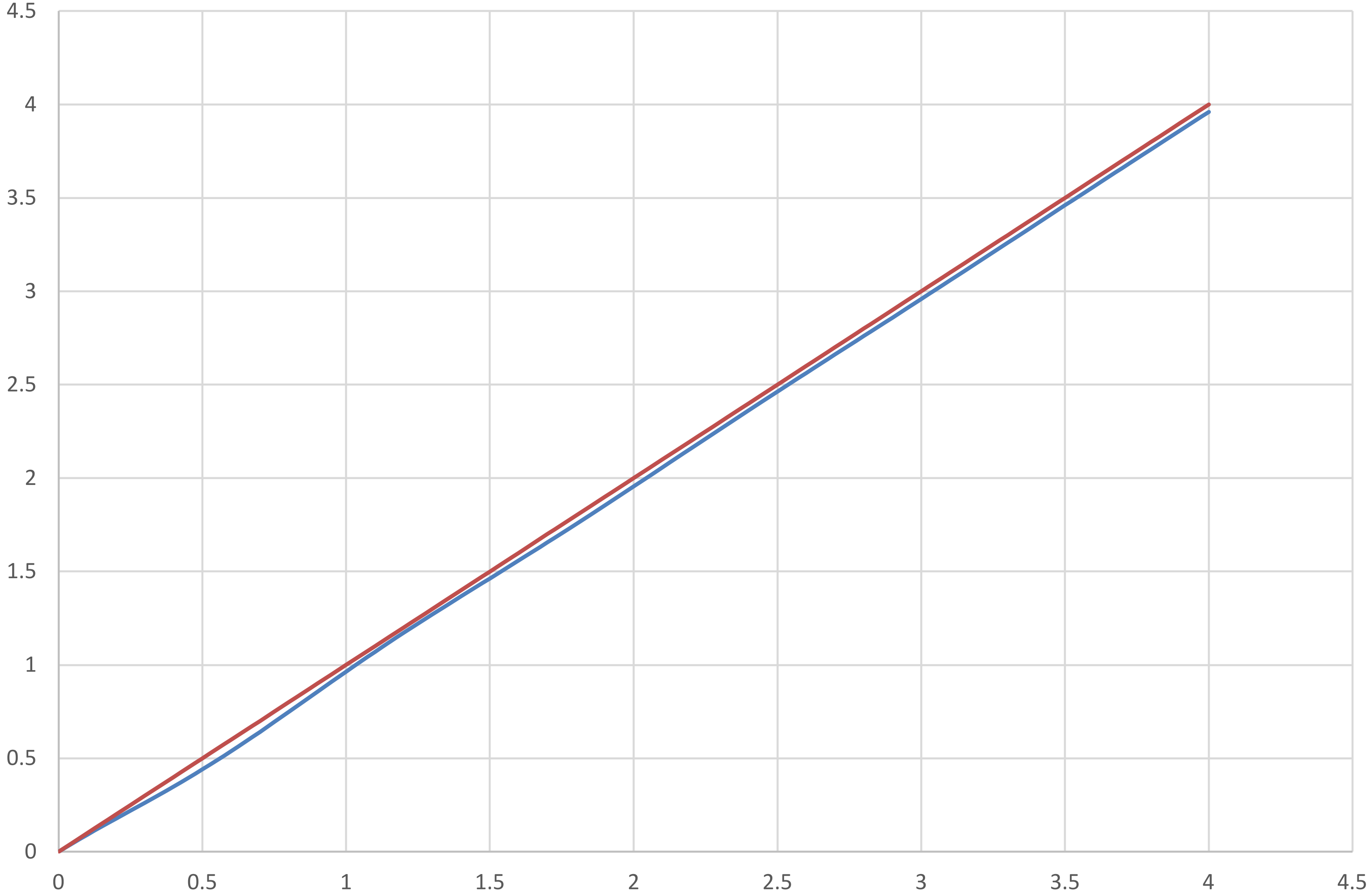
and from the b.d. $\qquad X_\mathrm{i}(s) = \frac{V_\mathrm{i}(s)}{K_4} = \frac{\Omega}{s^2 K_4} = \frac{\Omega_x}{s^2} \qquad (8)$

The output in s-domain $\qquad X_\mathrm{o}(s) = \frac{\omega_\mathrm{n}^2 \Omega_x}{s^2(s^2 + 2\gamma\omega_\mathrm{n} s + \omega_\mathrm{n}^2)} \qquad (9)$

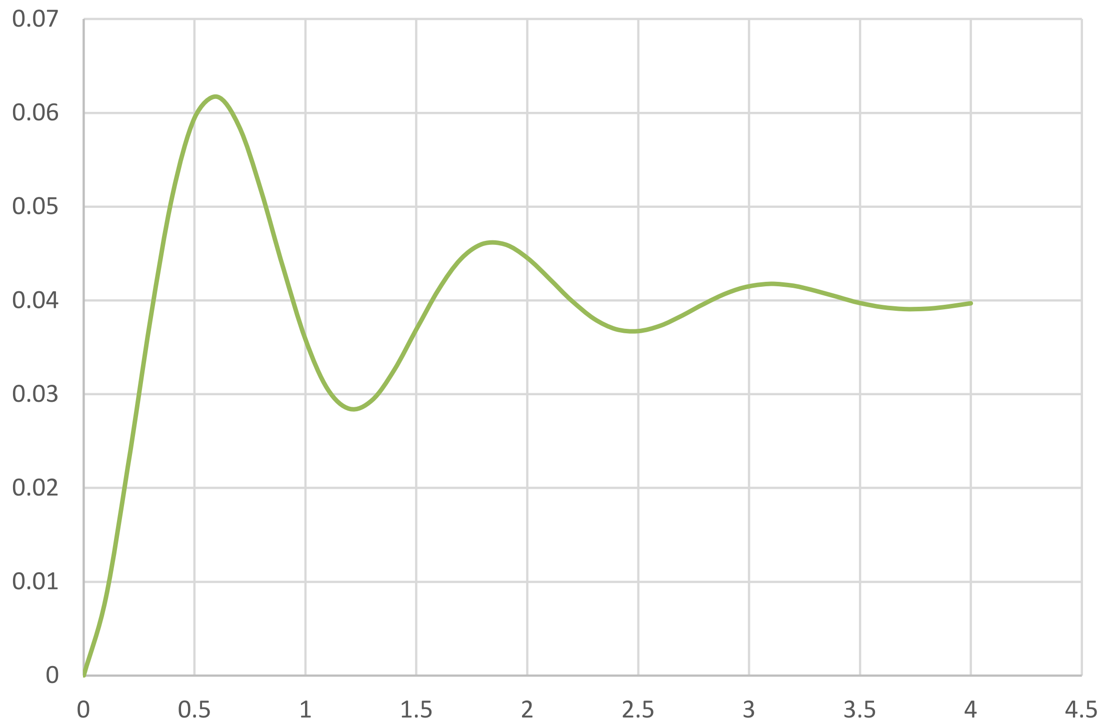
In the time domain

$$x_\mathrm{o}(t) = \Omega_x \left(t - \frac{2\gamma}{\omega_\mathrm{n}} + A_1 \mathrm{e}^{p_1 t} + A_2 \mathrm{e}^{p_2 t} \right) \qquad (10)$$

Output $x_o(t)$



e(t) for γ<1



E.-M. Position Control System: S.-S. Error under Ramp Input

From the block diagram, for $F_R = 0$ (no disturbance)

$$E(s) = X_i(s) - X_o(s) = \frac{Ms^2 + Cs}{Ms^2 + Cs + K} X_i(s) \qquad (11)$$

For a ramp input $X_i(s)$ from Eq. (8)

$$E(s) = \frac{Ms^2 + Cs}{Ms^2 + Cs + K} \frac{\Omega_x}{s^2} = \frac{Ms + C}{Ms^2 + Cs + K} \frac{\Omega_x}{s} \qquad (12)$$

Using the **final value theorem** the steady-state error

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{Ms + C}{Ms^2 + Cs + K} \Omega_x$$

$$= \frac{C}{K} \Omega_x = \frac{2\gamma}{\omega_n} \Omega_x \qquad (13)$$

What Next?

- improving transient and steady-state performance: velocity feedback & PID control
- stability of feedback systems