The University of Nottingham

SCHOOL OF MATHEMATICAL SCIENCES

A LEVEL 2 MODULE, AUTUMN SEMESTER 2020-2021

ADVANCED MATHEMATICS AND STATISTICS FOR ENGINEERS

Time to complete TWO Hours plus SIXTY Minutes upload time

Paper set: 13/01/2021 - 09:00

Paper due: 13/01/2021 - 12:00

Answer ALL questions

Your solutions should be written on white paper using dark ink (not pencil), on a tablet, or typeset. Do not write close to the margins. Your solutions should include complete explanations and all intermediate derivations. Your solutions should be based on the material covered in the module and its prerequisites only. Any notation used should be consistent with that in the Lecture Notes.

Submit your answers as a single PDF with each page in the correct orientation, to the appropriate dropbox on the module's Moodle page. Use the standard naming convention for your document: [StudentID]_[ModuleCode].pdf.

A scan of handwritten notes is completely acceptable. Make sure your PDF is easily readable and does not require magnification. Text which is not in focus or is not legible for any other reason will be ignored. If your scan is larger than 20Mb, please see if it can easily be reduced in size (e.g. scan in black & white, use a lower dpi — but not so low that readability is compromised).

Staff are not permitted to answer assessment or teaching queries during the assessment period. If you spot what you think may be an error on the exam paper, note this in your submission but answer the question as written. Where necessary, minor clarifications or general guidance may be posted on Moodle for all students to access.

Submissions received after the deadline will receive a mark of zero.

Academic Integrity in Alternative Assessments

Work submitted for assessment should be entirely your own work. You must not collude with others or engage the services of others (paid for or not) to work on your assignment. As with all assessments, you also need to avoid plagiarism. Plagiarism, collusion and false authorship are all examples of academic misconduct. They are defined in the University Academic Misconduct Policy at: https://www.nottingham.ac.uk/academicservices/qualitymanual/assessmentandawards/academic-misconduct.aspx

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Statement of Academic Integrity

By submitting a piece of work for assessment you are agreeing to the following statements:

1. I confirm that I have read and understood the definitions of plagiarism, false authorship and collusion.
2. I confirm that this assessment is my own work and is not copied from any other person's work (published or unpublished).
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4. I understand that plagiarism, false authorship, and collusion are academic offences and I may be referred to the Academic Misconduct Committee if plagiarism, false authorship or collusion is suspected.

1. (a) Find the general solution of the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = 13x^2 + 25x + 21.$$

[9 marks]

(b) The concentations of salt in two mixing tanks are governed by the equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -4x + 3y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 7 + x - 6y,\end{aligned}$$

subject to the initial conditions

$$x(0) = 1 \qquad \text{and} \qquad y(0) = 0.$$

Find $x(t)$ and $y(t)$.

[11 marks]

2. A function $f(x)$ is defined by the conditions

$$f(x+4) = f(x) \qquad \text{and} \qquad f(x) = \begin{cases} 0 & -2 \le x < -1 \\ x & -1 < x < 1 \\ 0 & 1 \le x < 2. \end{cases}$$

(a) Sketch a graph of this function for $-6 < x < 6$.

[3 marks]

(b) Find a general expression for the Fourier coefficients of $f(x)$ and write out explicitly the first four nonzero terms of the Fourier series.

[10 marks]

(c) What does the Fourier series converge to when (i) $x = 1$ and (ii) $x = 2$?

[2 marks]

(d) Find a particular integral of the differential equation

$$2\frac{\mathrm{d}y}{\mathrm{d}t} + \pi y = f(x).$$

in the form of a Fourier series and write out the first four nonzero terms explicitly.

[5 marks]

3. A function $f(t)$ is defined by the conditions

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < \infty. \end{cases}$$

(a) Sketch a graph of $f(t)$ and write it in terms of Heaviside step functions.

[4 marks]

(b) Show that

$$\bar{f}(s) = \frac{a + b\mathrm{e}^{-s}}{s^2}$$

where a and b are constants you should determine. [3 marks]

(c) Find the Laplace transform $\bar{y}(s)$ of the function satisfying the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 7\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = 300f(t)$$

subject to the initial conditions

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0.$$

[5 marks]

(d) Hence find $y(t)$.

[8 marks]

4. The temperature $\varphi(x,t)$ along an imperfectly insulated rod of length L is governed by the equation

$$\frac{\partial \varphi}{\partial t} = D\frac{\partial^2 \varphi}{\partial x^2} - a\varphi, \qquad \text{for} \quad 0 < x < L \quad \text{and} \quad t > 0, \tag{1}$$

where D and a are positive constants.

(a) Show that a separation of variables substitution $\varphi(x,t) = X(x)T(t)$ works if $X(x)$ and $T(t)$ respectively satisfy the differential equations

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda X = 0$$

and

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \lambda' T = 0,$$

where λ and λ' are constants, and state the relationship between λ and λ'.

[5 marks]

(b) By carefully considering the cases where λ is positive, zero or negative, find all solutions $X(x)$ consistent with $\varphi(x,t)$ satisfying the boundary conditions

$$\varphi(x=0,t) = 0 \qquad \text{and} \qquad \varphi_x(x=L,t) = 0, \qquad \text{for all} \quad t > 0, \tag{2}$$

where $\varphi_x = \partial\varphi/\partial x$.

[10 marks]

(c) Hence find a general solution of (1) consistent with the boundary conditions (2).

[5 marks]

5. (a) Events A, B and C respectively have probabilities $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{5}$. It is also known that $P(A \cap B) = \frac{1}{5}$, $P(B \cap C) = \frac{1}{10}$ and $P(C \cap A) = \frac{1}{10}$.

i) Determine the probability that C occurs, given that B occurs.

ii) Given that B occurs, which of A or C is more likely to occur?

iii) Are events A and C independent?

[6 marks]

(b) BTTF Industries buys 40% of its flux capacitors from Tesla and 60% of its flux capacitors from DMC. It is found that 1.5% of Tesla's flux capacitors fail within a year, whereas 0.5% of DMC's flux capacitors fail over the same time.

i) What percentage of BTTF Industries' total flux capacitor supply fails within a year?

ii) An order is placed for 50 flux capacitors from DMC. It is known that the number of flux capacitors failing within a year follows a binomial distribution. What is the probability that no more than two of the flux capacitors in this batch fail within a year?

[5 marks]

(c) The resistances of a sample of 100 electrical components were tested in order to determine whether a batch is within specification, which requires that the components should have resistances between 45Ω and 55Ω (Ohms). The measurements yielded

$$\sum_{i=1}^{100} R_i = 5123\,\Omega$$

and

$$\sum_{i=1}^{100} R_i^2 = 262,923\,\Omega^2.$$

i) Calculate the sample mean $\bar{R}$ and the sample variance s^2 from this test.

ii) Assuming a normal distribution, and using the calculated sample mean and variance, calculate the probability that a component in this batch does not meet specifications for resistance.

iii) Construct a 95% confidence interval for the mean resistance calculated from the sample, and comment on what this implies about your calculations in part ii).

[9 marks]