MTHS2007 2020/21 Exam solutions.

Q1 (a) Auxiliary equation:

$$0 = m^2 + 6m + 13 = (m+3)^2 + 4$$
$$\Rightarrow m = -3 \pm 2i$$

so the complementary function is

$$y_c(x) = e^{-3x}(A\cos 2x + B\sin 2x).$$

For P.I. try $y_p = ax^2 + bx + c$

$$\Rightarrow y_p' = 2ax + b$$
$$y_p'' = 2a$$

$$\Rightarrow y_p'' + 6y_p' + 13y_p = 2a + 12ax + 6b + 13(ax^2 + bx + c)$$
$$= 13ax^2 + (13b + 12a)x + 13c + 2b + 2a$$
$$= 13x^2 + 25x + 21$$

if

$$\left.\begin{aligned} 13a &= 13 \\ 13b + 12a &= 25 \\ 13c + 6b + 2a &= 21 \end{aligned}\right\} \Rightarrow \begin{aligned} &a = 1 \\ &13b = 25 - 12 \Rightarrow b = 1 \\ &13c = 21 - 6 - 2 \Rightarrow c = 1 \end{aligned}$$

$$\Rightarrow \quad y_p(x) = x^2 + x + 1$$

$$\Rightarrow \quad y(x) = e^{-3x}(A\cos 2x + B\sin 2x) + x^2 + x + 1$$

(b) $\frac{dx}{dt} = -4x+3y \Rightarrow \frac{d^2x}{dt^2} = -4\frac{dx}{dt} + 3\frac{dy}{dt} = -4\frac{dx}{dt} + 3(7+x-6y)$

$$= -4\frac{dx}{dt} + 21 + 3x - 6\left(\frac{dx}{dt} + 4x\right)$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 21x = 21$$

Auxiliary equation $\quad 0 = m^2 + 10m + 21 = (m+3)(m+7)$

$\Rightarrow \quad m = -3, -7$

Complementary fn $\quad x_c(t) = Ae^{-3t} + Be^{-7t}$

For P.I. try $x_p = a \Rightarrow 21a = 21 \Rightarrow a = 1 \Rightarrow x_p(t) = 1$

$$\Rightarrow \quad x(t) = 1 + Ae^{-3t} + Be^{-7t}$$

$$3y(t) = x'(t) + 4x$$
$$= 4 + Ae^{-3t} - 3Be^{-7t}$$

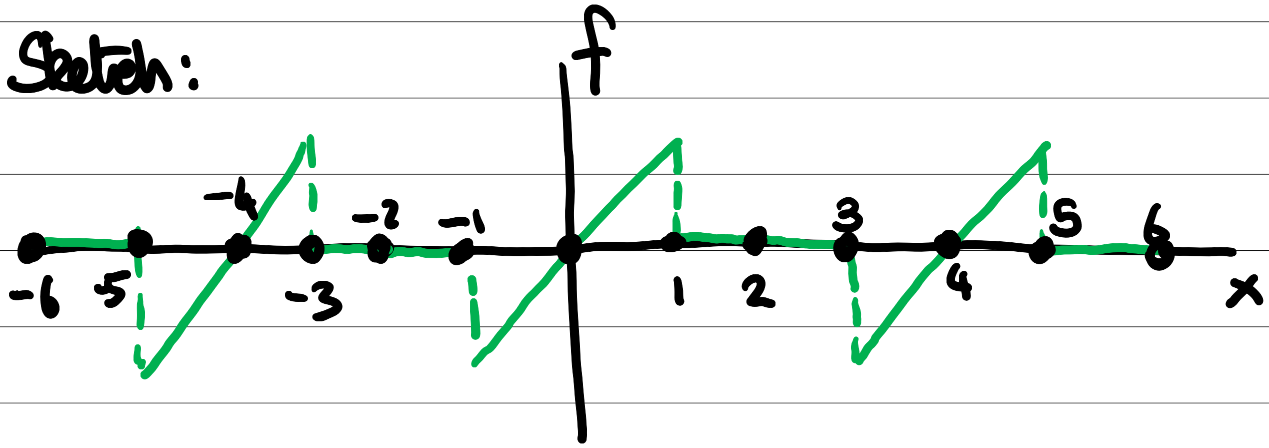
$\Rightarrow$ general solution $\quad x(t) = 1 + Ae^{-3t} + Be^{-7t}$

$y(t) = \frac{4}{3} + \frac{1}{3}Ae^{-3t} - Be^{-7t}$

i.c's $\left.\begin{array}{l} x(0) = 1 = 1 + A + B \\ y(0) = 0 = \frac{4}{3} + \frac{1}{3}A - B \end{array}\right\} \Rightarrow \begin{array}{l} A = -1 \\ B = 1 \end{array}$

$$\Rightarrow \quad x(t) = 1 - e^{-3t} + e^{-7t}$$
$$y(t) = 4/3 - 1/3e^{-3t} - e^{-7t}$$

(a) Sketch:



(b) $f(x)$ is an odd function, half period $L=2$ so

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$b_n = \frac{1}{2}\int_{-2}^{2} \sin\left(\frac{n\pi x}{2}\right) f(x)\,dx$$

$$= \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 x\,\frac{d}{dx}\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right)dx$$

$$= -\frac{2}{n\pi}\left[x\cos\frac{n\pi x}{2}\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{n\pi x}{2}\,dx$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n\pi}\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_0^1$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} \dfrac{4}{n^2\pi^2}\sin\dfrac{n\pi}{2} & n \text{ odd} \\ -\dfrac{2}{n\pi}\cos\dfrac{n\pi}{2} & n \text{ even} \end{cases}$$

$$\Rightarrow f(x) = \frac{4}{\pi^2}\left(\sin\frac{\pi x}{2} - \frac{\sin\frac{3\pi x}{2}}{9} + \dots \right)$$

$$+ \frac{1}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x - \dots \right)$$

(c)(i) f is discontinuous at $x=1$: $f(1^-)=1$, $f(1^+)=0$, so F.S. converges to $\frac{1}{2}(1+0) = \frac{1}{2}$.

(ii) f is continuous at $x=2$: F.S. converges to $f(2)=0$.

(d) Let $y(x) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{2} + B_n \sin\frac{n\pi x}{2}$

$$\Rightarrow y'(x) = \frac{\pi}{2}\sum_{n=1}^{\infty} nB_n \cos\frac{n\pi x}{2} - nA_n \sin\frac{n\pi x}{2}$$

$$\Rightarrow 2y' + \pi y = \frac{\pi A_0}{2} + \pi\sum_{n=1}^{\infty} (A_n + nB_n)\cos\frac{n\pi x}{2}$$

$$+ (B_n - nA_n)\sin\frac{n\pi x}{2}$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

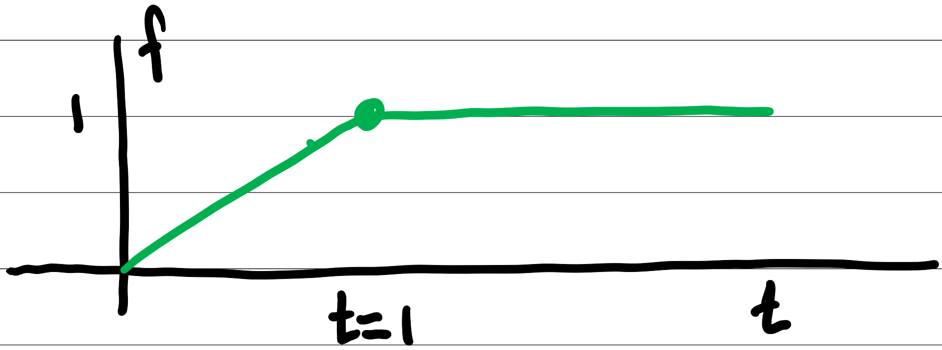
if $A_0 = 0$, $\left.\begin{aligned} A_n + nB_n &= 0 \\ B_n - nA_n &= b_n/\pi \end{aligned}\right\} \Rightarrow \begin{aligned} A_n &= -nB_n \\ (1+n^2)B_n &= b_n/\pi \end{aligned}$

$$\Rightarrow B_n = \frac{b_n}{\pi(1+n^2)}, \quad A_n = \frac{-nb_n}{\pi(1+n^2)}$$

$$\Rightarrow y(x) = -\frac{1}{2\pi}\frac{4}{\pi^2}\cos\frac{\pi x}{2} + \frac{1}{2\pi}\frac{4}{\pi^2}\sin\frac{\pi x}{2}$$

$$- \frac{2}{5\pi}\frac{1}{\pi}\cos\pi x + \frac{1}{5\pi}\frac{1}{\pi}\sin\pi x + \dots$$

Q3 (a)



$$f(t) = t\,(1 - H(t-1)) + H(t-1)$$
$$= t - (t-1)\,H(t-1)$$

(b)

$$\bar{f}(s) = \frac{1}{s^2} - \mathcal{L}\big((t-1)\,H(t-1)\big)$$
$$= \frac{1}{s^2} - e^{-s}/s^2 \quad \text{by second shifting theorem}$$
$$(a=1,\ b=-1)$$

(c) $y'' + 7y' + 10y' = 300f$

$$\xrightarrow{\text{L.T.}} \quad s^2\bar{y} + 7s\bar{y} + 10\bar{y} = 300\bar{f}(s)$$
$$= 300\left(\frac{1}{s^2} - e^{-s}/s^2\right)$$

$$\Rightarrow \bar{y}(s) = 300\,\frac{1-e^{-s}}{s^2(s^2+7s+10)}$$

(d)

$$\frac{300}{s^2(s^2+7s+10)} = \frac{300}{s^2(s+2)(s+5)}$$
$$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+5}$$

$$= \frac{A\,s(s+2)(s+5)+B(s+2)(s+5)+C s^2(s+5)+D s^2(s+2)}{s^2(s+2)(s+5)}$$

$$\Rightarrow A\,s(s+2)(s+5)+B(s+2)(s+5)+C\,s^2(s+5)+D s^2(s+2)=300$$

$$\left.\begin{array}{l} s=0 \Rightarrow 10B=300 \\ s=-2 \Rightarrow 12C=300 \\ s=-5 \Rightarrow -75D=300 \end{array}\right\} \Rightarrow \begin{array}{l} B=30 \\ C=\frac{300}{12}=25 \\ D=-\frac{300}{75}=-4 \end{array}$$

$$s=-1 \Rightarrow -4A+4B+4C+D=300$$

$$\Rightarrow -4A=300-D-4C-4B$$

$$=300+4-100-120=84$$

$$\Rightarrow A=-21$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{300}{s^2(s^2+7s+10)}\right)=\mathcal{L}^{-1}\left(-\frac{21}{s}+\frac{30}{s^2}+\frac{25}{s+2}-\frac{4}{s+5}\right)$$

$$=-21+30t+25e^{-2t}-4e^{-5t}$$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2(s^2+7s+10)}\right)=H(t-1)\Big[-21+30(t-1)+25e^{-2(t-1)}$$
$$-4e^{-5(t-1)}\Big]$$

$$\Rightarrow y(t)=-21+30t+25e^{-2t}-4e^{-5t}$$
$$-H(t-1)\Big[-21+30(t-1)+25e^{-2(t-1)}-4e^{-5(t-1)}\Big]$$

Q4 (a) $\varphi = XT \Rightarrow XT' = DX''T - aXT$

$$\Rightarrow \frac{T'}{T} = D\frac{X''}{X} - a$$

$$\Rightarrow D\frac{X''}{X} = T'/T + a$$

Each side must equal (the same) constant: α

$$D\frac{X''}{X} = \alpha \Rightarrow X'' - \frac{\alpha}{D}X = 0$$

$$\Rightarrow X'' + \lambda X = 0 \quad \text{if } \lambda = -\alpha/D$$

$$T'/T + a = \alpha = -D\lambda \Rightarrow T' = -(a + D\lambda)T$$

$$T' + \lambda' T = 0, \; \lambda' = a + D\lambda$$

(b) • If $\lambda = -m^2 < 0 \quad X(x) = Ae^{mx} + Be^{-mx}$

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow X(x) = A(e^{mx} - e^{-mx})$$

Then $X'(L) = mA(e^{mL} + e^{-mL}) = 0 \Rightarrow A = 0$
so there is only the trivial solution in this case.

• If $\lambda = 0 \quad X(x) = A + Bx$ and $X(0) = 0 \Rightarrow A = 0$

$$X'(L) = 0 \Rightarrow B = 0$$

so once again only trivial solution.

• If $\lambda = m^2 > 0$ then $X(x) = A\cos mx + B\sin mx$

$X(0)=0 \Rightarrow A=0$ and then $X(x)=B\sin mx$

$X'(L)=0 \Rightarrow B\cos mL=0 \Rightarrow \cos mL=0$ for non trivial solutions. Then

$$mL = \frac{n\pi}{2} \qquad n = \text{odd integer}$$

$$\Rightarrow \qquad X(x) = \sin\frac{n\pi x}{2L} \qquad n=1,3,5,\ldots$$

(c) If $m=\frac{n\pi}{2L}$, $\lambda=m^2=\frac{n^2\pi^2}{4L^2}$, $\lambda' = a+D\lambda$
$= a+\frac{n^2\pi^2 D}{4L^2}$

$$T'+\lambda' T=0 \Rightarrow T = Ce^{-\lambda' t} = Ce^{-\left(a+\frac{n^2\pi^2 D}{4L^2}\right)t}$$

So the general solution is

$$\varphi(x,t) = \sum_{n\text{ odd}} C_n e^{-\left(a+\frac{n^2\pi^2 D}{4L^2}\right)t}\sin\frac{n\pi x}{2L}$$

Q5(a)(i) $P(C|B) = P(C \cap B) / P(B)$

$= 1/10 \, / \, 1/4$

$= 4/10 = 2/5$

(ii) $P(A|B) = P(A \cap B) / P(B)$

$= 1/5 \, / \, 1/4$

$= 4/5$

$\Rightarrow$ A is more likely

(iii) A, C are independent if

$$P(A \cap C) = P(A)\,P(C)$$

$P(A \cap C) = \frac{1}{10}$ $\quad P(A) \cdot P(C) = \frac{1}{2} \frac{1}{5}$

which are equal, so it is true.

(b) (i) Assuming failure is independent,

$P(\text{failure}) = P(\text{Tesla})\, P(\text{Tesla failure}) + P(\text{DMC})\, P(\text{DMC failure})$

$= 0.4 \; 1.5\% + 0.6(0.5\%)$

$= 0.9\%$

(iii) Use $\text{bin}(n, p)$ with $n = 50$ and $p = 0.005$

$q = 1 - p = 0.995$

$P(X=0)+P(X=1)+P(X=2)$

$$= q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2$$

$$= (0.995)^{50} + 50(0.995)^{49} \cdot 0.005 + \frac{50 \cdot 49}{2}(0.995)^{48}(0.005)^2$$

$$= 0.7783 + 0.1956 + 0.0241 = 0.9980$$

(c) (i) $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i = \frac{5123}{100}\,\Omega = 51.23\,\Omega$

$$s^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} R_i^2 - n\bar{R}^2\right) = \frac{1}{99}(262,923-262,451)\,\Omega^2$$
$$= 4.77\,\Omega^2$$

$\Rightarrow s = 2.18\,\Omega$

(ii) For probability that $R > 55\,\Omega$ use $z = \frac{R-\bar{R}}{s}$
$= (55-51.23)/2.18 = 1.73$ in $N(0,1)$ to get

$$P(R>55) = 1-F(1.73) = 1-0.9582 = 0.0418$$

For probability that $R < 45$ use $z = \frac{45-51.23}{2.18} = -2.86$
to get

$$P(R<45) = 1-F(2.86) = 1-0.9979 = 0.0021$$

Therefore prob that component out of spec is

$$P(R>55)+P(R<45) = 0.0418+0.0021 = 0.0439$$

(iii) Here $z = \dfrac{\bar{R} - \mu}{\sigma/\sqrt{n}} \simeq \dfrac{\bar{R} - \mu}{s/10}$ has a 95% confidence interval

$$-1.96 < \frac{\bar{R} - \mu}{s/10} < 1.96$$

$$\Rightarrow \quad \bar{R} - 1.96\frac{s}{10} < \mu < \bar{R} + 1.96\frac{s}{10}$$

$$\Rightarrow \quad 51.23 - 0.43 < \mu < 51.23 + 0.43$$

Uncertainty in mean value is a significant proportion of sample variance $s = 2.18\,\Omega$ calculated in (ii) so probabilities calculated may be significantly changed when new samples are taken, although the main conclusion that components are likely to be in spec remain valid.