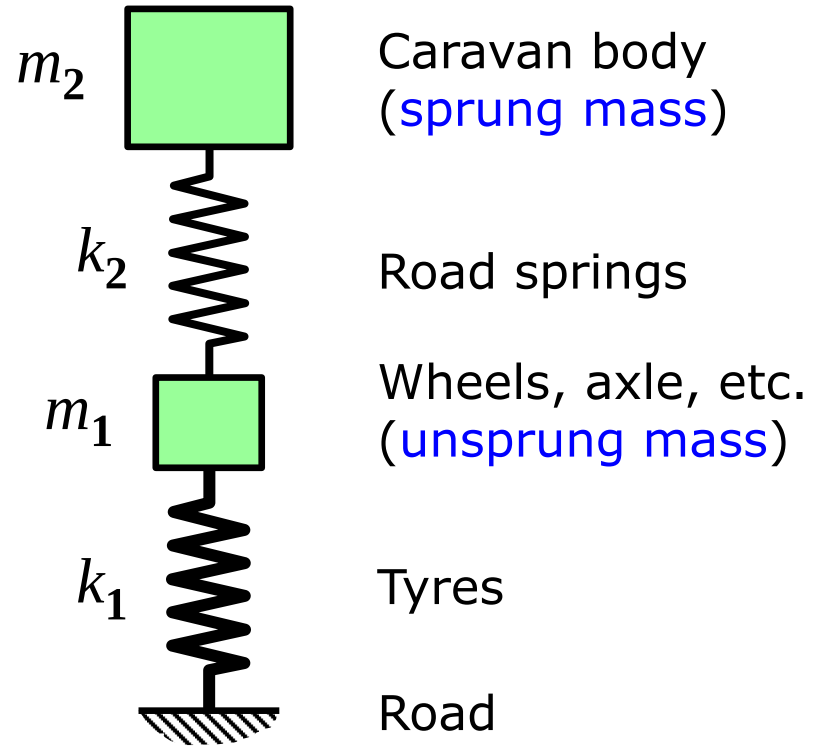
MULTI-DEGREE-OF-FREEDOM SYSTEMS

Model of a single-axle caravan and its suspension



The body and the axle can move separately, so the model has **2 degrees-of-freedom** (2 independent *possible motions*)

We need 2 coordinates to describe the motion

– **axle displacement** and **body displacement**

System has **2 natural frequencies**, each with a characteristic pattern of displacement, called a ***MODE SHAPE***

Definition of Mode Shape

Characteristic deflection pattern for a structure when it vibrates at one of its natural frequencies

We will study 2 classes of structures that have more than one mode of vibration

- **Multi-degree-of-freedom systems** which have discrete masses and springs
- **Shafts and beams** which have distributed mass and stiffness

In both cases, the aim will be to calculate

1. The **natural frequencies** of the system
2. The **corresponding mode shapes**

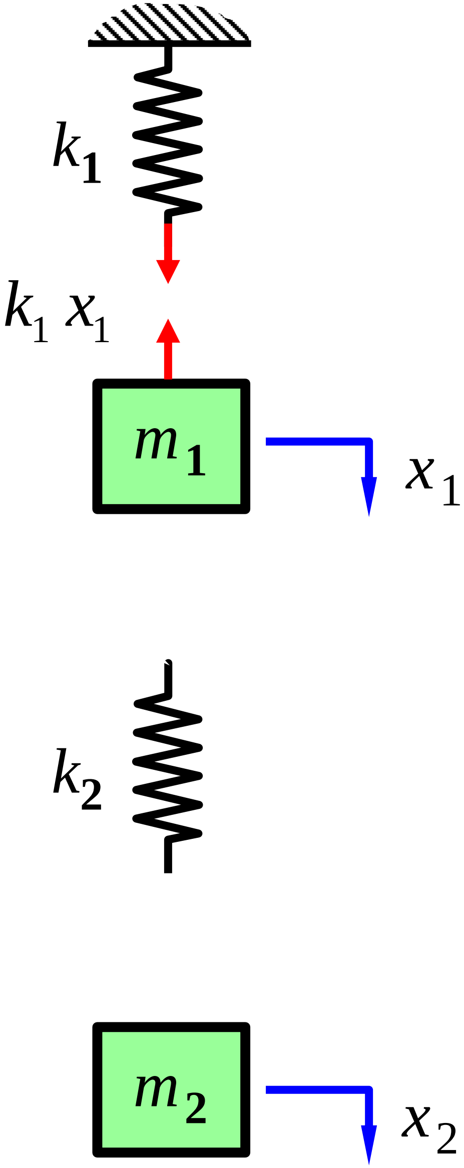
Example 1: Demonstration System (2 Degrees of Freedom)

STEP 2: Free-body Diagrams

(i) Remove the springs from the masses

(ii) Mark the chosen positive directions for motion

(iii) For positive displacements, write down the expressions for the forces and add them to the diagram to show their positive direction



Force in spring k_2 ?

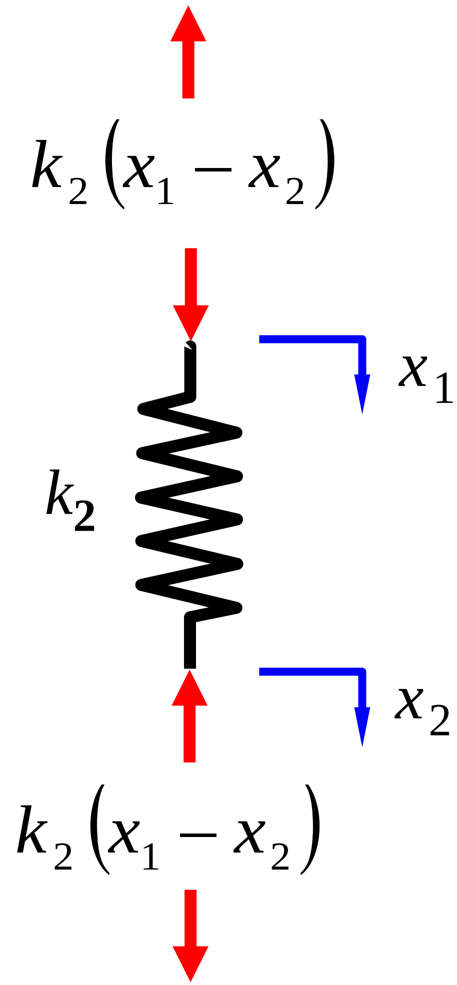
Force in spring k_2 ?

The expression for the force in a spring is

Spring force = Stiffness x Change of length

What is the **change of length** of the spring?

Is the spring in **tension** or **compression**?



If both ends of the spring **move down**, the **net change of length** is the **difference** in displacements

That is $(x_1 - x_2)$

The force in the spring is $k_2(x_1 - x_2)$

Is the spring in **tension** or **compression**?

If $(x_1 - x_2)$ is positive, the spring is in **compression**

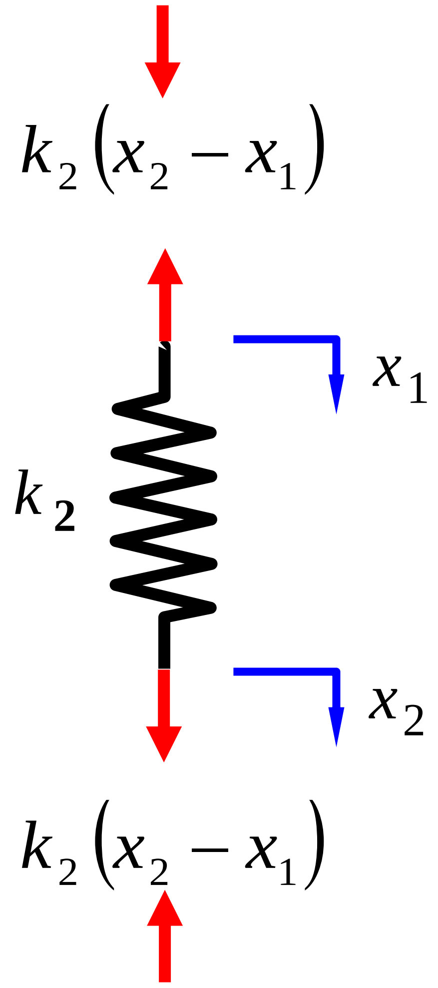
Force in spring k_2 ?

The expression for the force in a spring is

Spring force = Stiffness x Change of length

What is the **change of length** of the spring?

Is the spring in **tension** or **compression**?



An alternative expression for the **net change of length** is

$$\left(x_2 - x_1\right)$$

So the force in the spring is $k_2\left(x_2 - x_1\right)$

Is the spring in **tension** or **compression**?

If $\left(x_2 - x_1\right)$ is positive, the spring is in **TENSION**

Force in spring k_2 ?

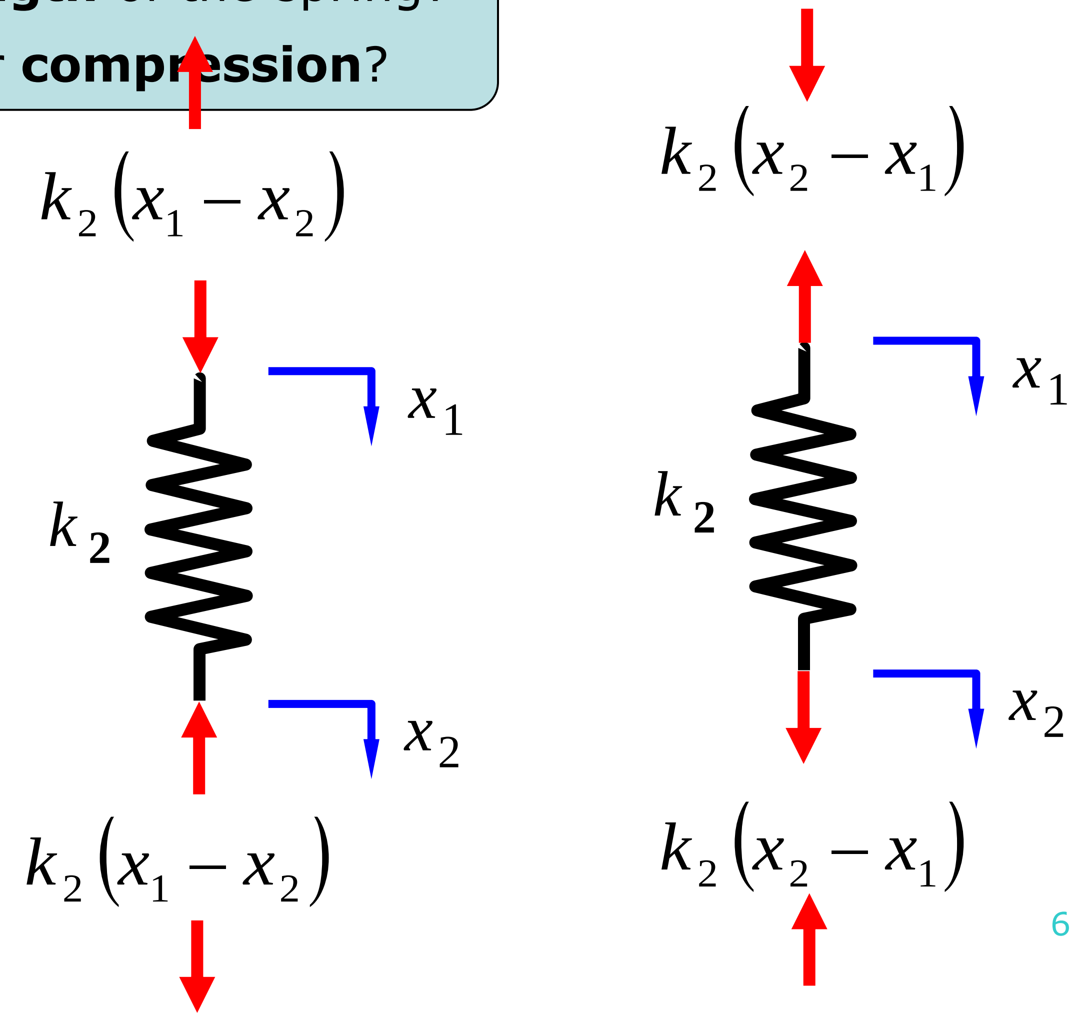
The expression for the force in a spring is

Spring force = Stiffness x Change of length

What is the **change of length** of the spring?

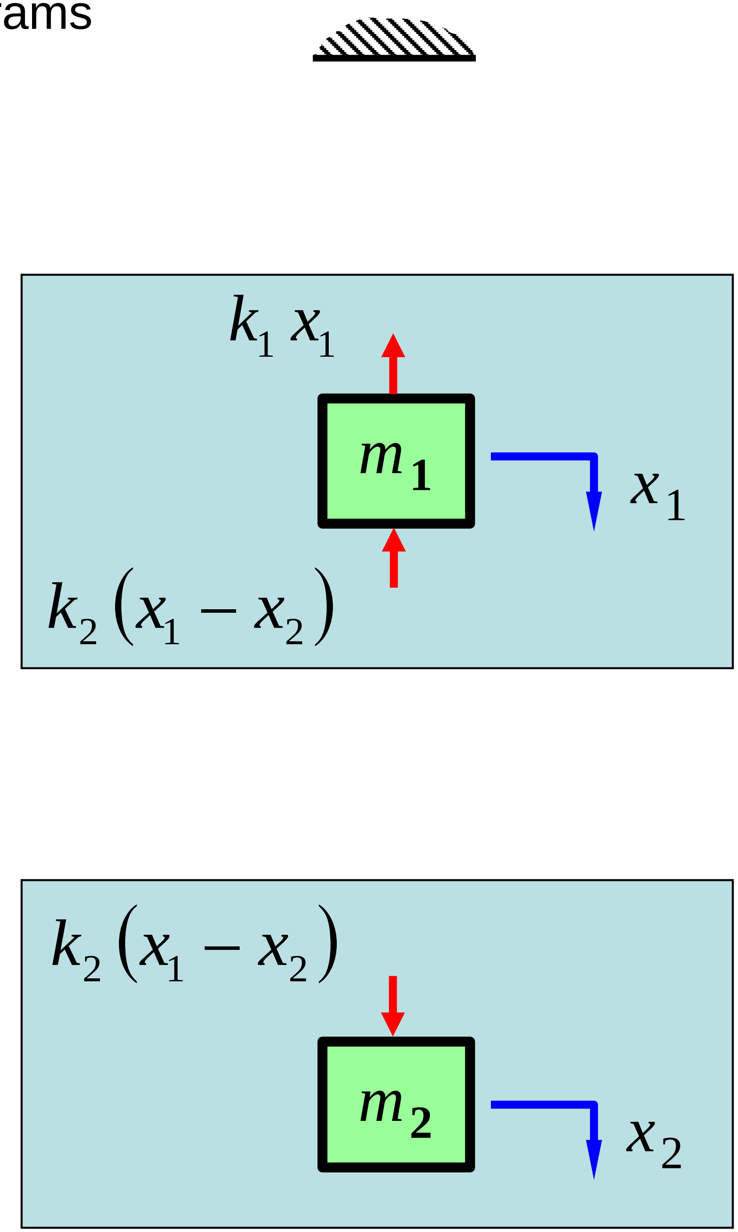
Is the spring in **tension** or **compression**?

The two are exactly equivalent



Example 1: Demonstration System (2 Degrees of Freedom)

STEP 2: Free-body Diagrams

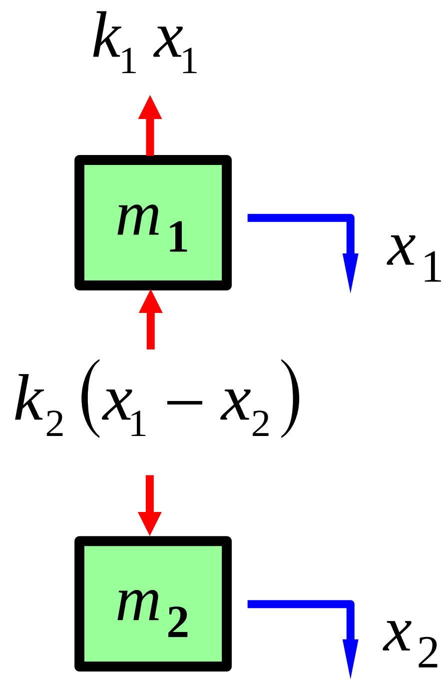


Completed free body diagrams

STEP 3: Equations of motion

$\downarrow x_1 \qquad -k_1 x_1 - k_2 (x_1 - x_2) = m_1 \ddot{x}_1$

$\downarrow x_2 \qquad +k_2 (x_1 - x_2) = m_2 \ddot{x}_2$



or

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \qquad (1a)$$

$$m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \qquad (1b)$$

In matrix form (& remembering the general formulation $M\ddot{z} + Kz = 0$)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

In matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or $$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

As with single-degree-of-freedom systems, we can check for errors in the equations at this stage.

In particular,

(1) the terms in the leading diagonals of $[M]$ and $[K]$ are **always positive**

(2) the off-diagonal terms may be positive **or** negative

(3) $[M]$ and $[K]$ are **often symmetric** about the leading diagonal

Solution to obtain **natural frequencies** and **mode shapes**

For free vibration of the system at one of its natural frequencies, the motion of each mass will be sinusoidal.

Use as substitutions, $x_1(t) = X_1 \cos\omega t$ and $x_2(t) = X_2 \cos\omega t$

So that $\ddot{x}_1 = -\omega^2 X_1 \cos\omega t$ $\ddot{x}_2 = -\omega^2 X_2 \cos\omega t$

Substituting into (1) and cancelling the common factor $\cos\omega t$

(1a) => $-\omega^2 m_1 X_1 + (k_1 + k_2) X_1 - k_2 X_2 = 0$

(1b) => $-\omega^2 m_2 X_2 - k_2 X_1 + k_2 X_2 = 0$

In matrix form

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(2a)

(2b)

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} (2a) \\ (2b) \end{matrix}$$

or

$$\left([K] - \omega^2 [M] \right) \{X\} = \{0\} \qquad (3)$$

or

$$[Z]\{X\} = \{0\}$$

where

$$[Z] = [K] - \omega^2 [M]$$

Equation (3) is an **eigenvalue problem**; normally presented in maths books as

$$\left([A] - \lambda [B] \right) \{X\} = \{0\}$$

The **eigenvalues** give the **natural frequencies** and the **eigenvectors** give the corresponding **mode shapes**

Obtaining the Natural Frequencies

For a non-trivial solution of equation (3) $\det\left[Z\right] = 0$

From Wikipedia for a 2x2 matrix… $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$

where $a = k_1 + k_2 - \omega^2 m_1$, $b = -k_2$, $c = -k_2$, and $d = k_2 - \omega^2 m_2$

Multiplying out the determinant gives

$$m_1 m_2 \omega^4 - \left(m_1 k_2 + m_2\left(k_1 + k_2\right)\right)\omega^2 + k_1 k_2 = 0 \quad (4)$$

Equation (4) is called the **Frequency Equation**, and you want to solve for ω.

Solving involves a standard quadratic equation

$$Ax^2 + Bx + C = 0$$

Where $x = \omega^2$, $A = m_2 m_2$, $B = \left(m_1 k_2 + m_2(k_1 + k_2)\right)$, and $C = k_1 k_2$

It will therefore have two roots , $x = \omega_{n1}^2$ and $x = \omega_{n2}^2$, where ω_{n1} and ω_{n2} are the two natural frequencies of the system

Obtaining the Mode Shapes

To find the corresponding mode shapes, we will substitute the frequency (ω) that we are interested in back into equations (2a) and (2b) to get the relationship between $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} (2a) \\ (2b) \end{matrix}$$

This will define the **relative movement of the masses**, which is what we mean by the **"Mode Shape".** I.e. it will tell us how much X_1 moves if X_2 moves a certain amount, or visa versa.

(2a) and (2b) are a pair of homogeneous equations, so we cannot find unique solutions for $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ separately

One way to solve this is to **give one value an amplitude of UNITY** and then **find the amplitude of the other relative to this**

Give one value an amplitude of unity and then **find the amplitude of the other relative to this**

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} (2a) \\ (2b) \end{matrix}$$

Let $X_2 = 1$. Equation (2b) gives

$$-k_2 X_1$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{k_2 - \omega^2 m_2}{k_2} \\ 1.0 \end{Bmatrix} \qquad (5)$$

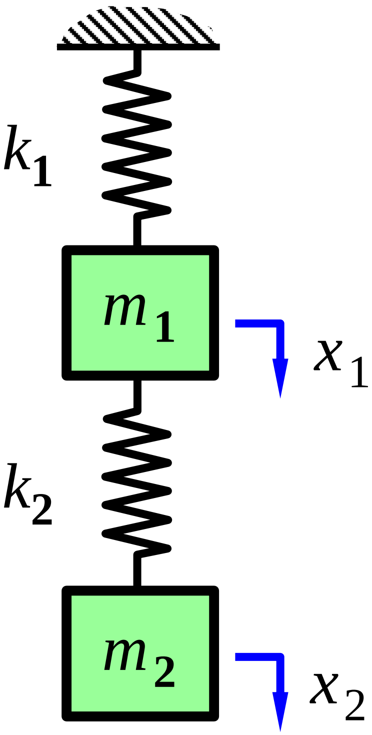
The vector $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$ is the required **Mode Shape** expression

What is reasonable to expect on the exam?

- In the exam you are expected to be able to
  - Form the generalized matrices ([Z]{X}={0}) for **any** DOF system
  - Solve for values of ω and {X} up to 2 DOF by hand
  - You are also expected to be able to sketch basic mode shapes given boundary conditions, without solving for values.

- I do not expect that you will solve by hand for numerical values of ω and {X} above 2DOF. But you should be aware of the techniques involved as future modules may expect you to do this.
  - These days though most people use Matlab or similar programs to solve more complex systems. But if you can not do the first step above then you have no hope of even utilizing those.

Example 1: Demonstration System



Objective is to find the natural frequencies and mode shapes of the system

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} (2a) \\ (2b) \end{matrix}$$

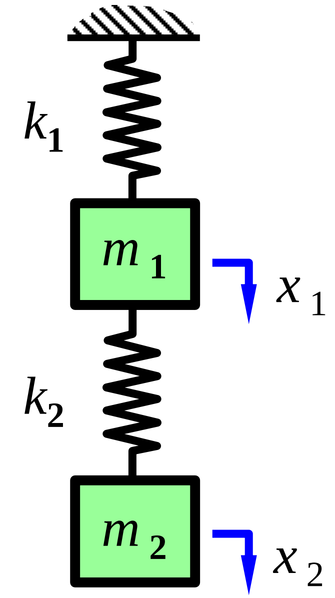
or $$[Z]\{X\} = \{0\}$$

Roots of the Frequency Equation $\det[Z] = 0$ give the natural frequencies

Equation (2b) gives the mode shape as

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{k_2 - \omega^2 m_2}{k_2} \\ 1.0 \end{Bmatrix} \qquad (5)$$

Numerical example $m_1 = m_2 = 2$ kg $k_1 = k_2 = 200$ N/m



Natural Frequencies

Substituting into (4) and solving gives $\omega_{n1}^2 = 38.1 \ \text{s}^{-2}$

and $\omega_{n2}^2 = 261.8 \ \text{s}^{-2}$

Hence, $\omega_{n1} = 6.18$ rad/s $= 0.98$ Hz

and $\omega_{n2} = 16.18$ rad/s $= 2.58$ Hz

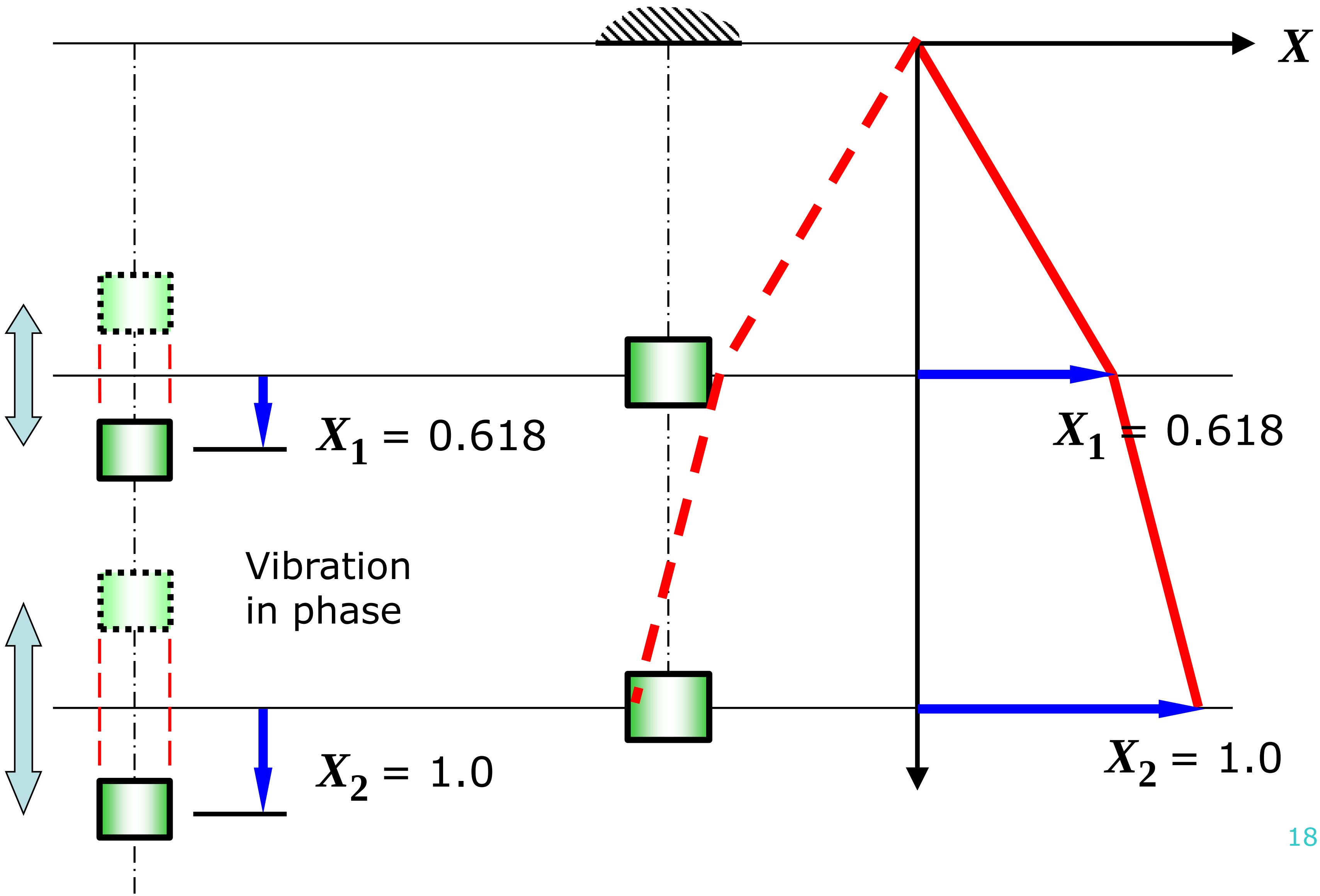
Mode Shapes

Mode #1 Put $\omega_{n1}^2 = 38.1 \ \text{s}^{-2}$ into (5) to give $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1.0 \end{Bmatrix}$

Mode #2 Put $\omega_{n2}^2 = 261.8 \ \text{s}^{-2}$ into (5) to give $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} -1.618 \\ 1.0 \end{Bmatrix}$

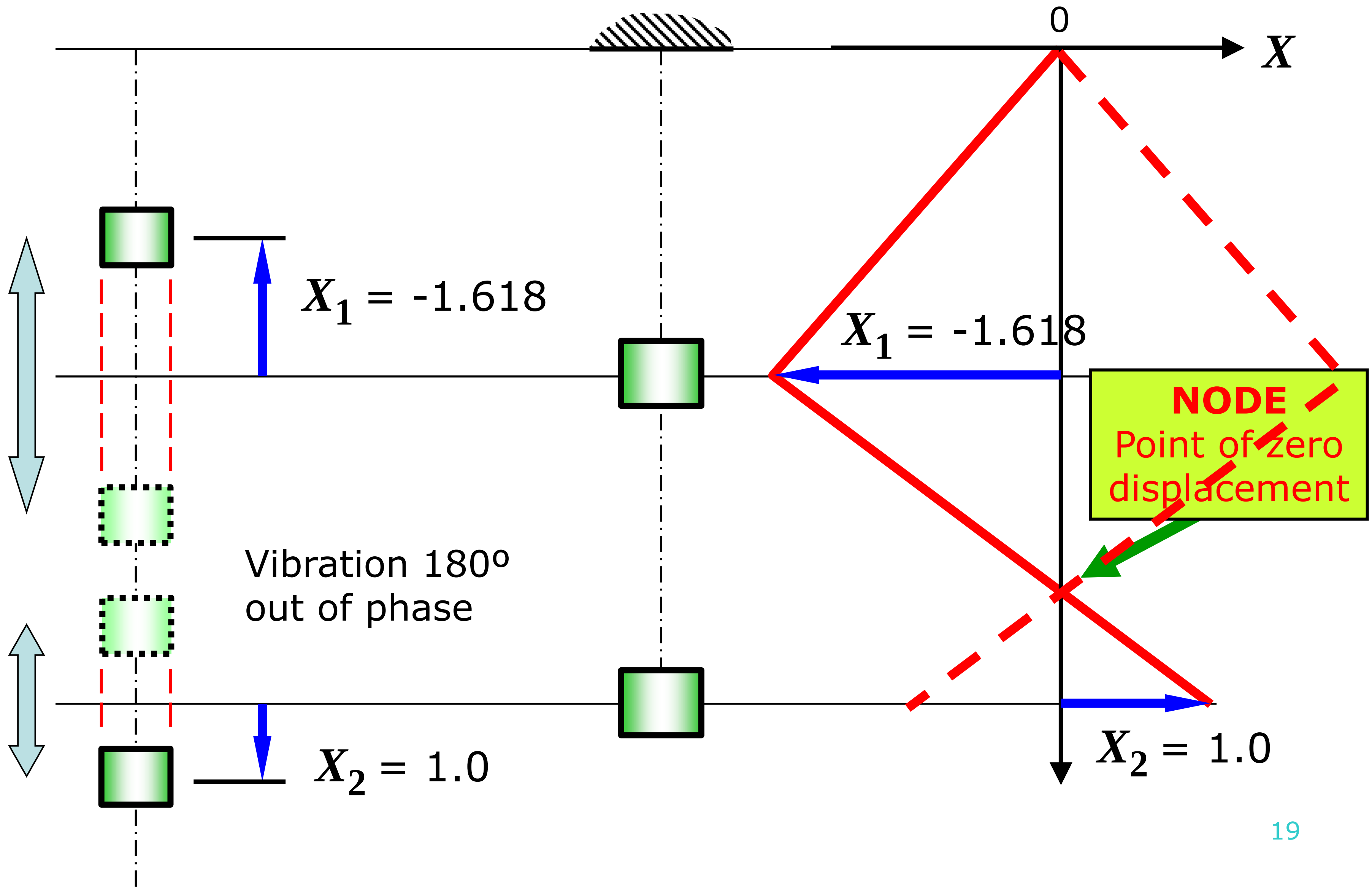
Mode Shape for Mode #1

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1.0 \end{Bmatrix}$$



Mode Shape for Mode #2

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} -1.618 \\ 1.0 \end{Bmatrix}$$

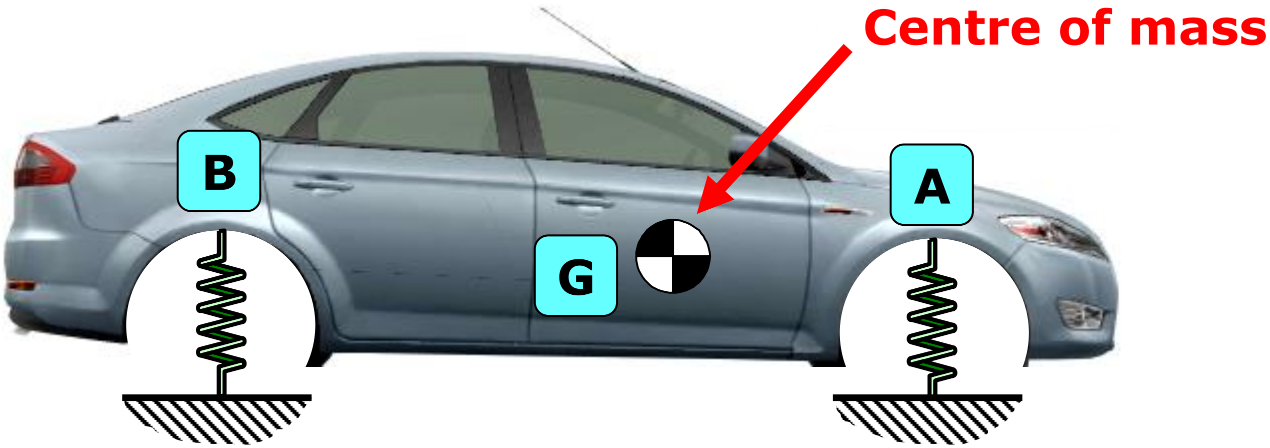


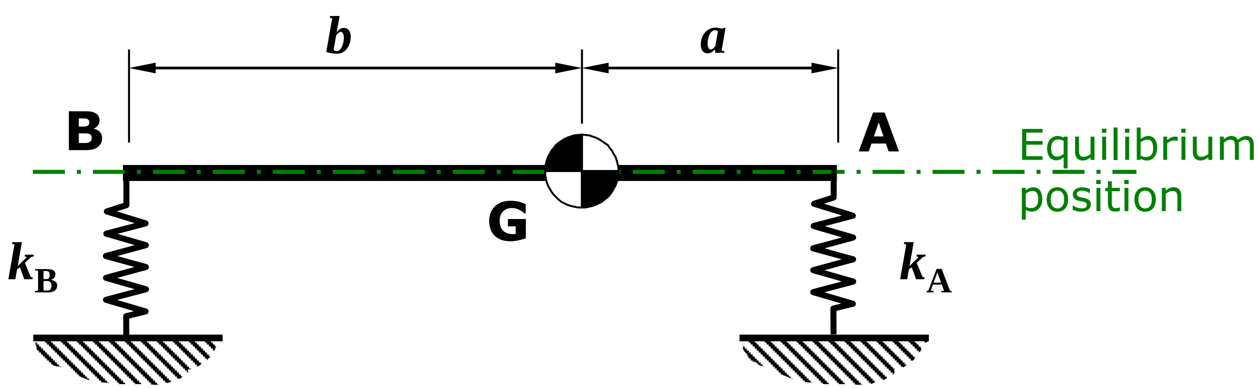
Example 2 **2D vehicle model**
(coupled bounce and pitch)

STEP 1 Assumptions used to develop the dynamic model

1. There is **no roll motion** - pitch and vertical translation only
2. The **body is rigid**, with mass, m, and moment of inertia, I_G
3. The **tyres are very stiff** so that the axles do not move
4. k_A and k_B are the combined stiffnesses for the front and rear springs respectively
5. **No shock absorbers** (dampers)

STEP 1 Dynamic mass-spring model





Selection of Coordinates

We could use x_G (displacement of G) together with θ (pitch angle) or we could use the displacements of A and B, x_A and x_B

Q *Does it matter which pair we use ?*

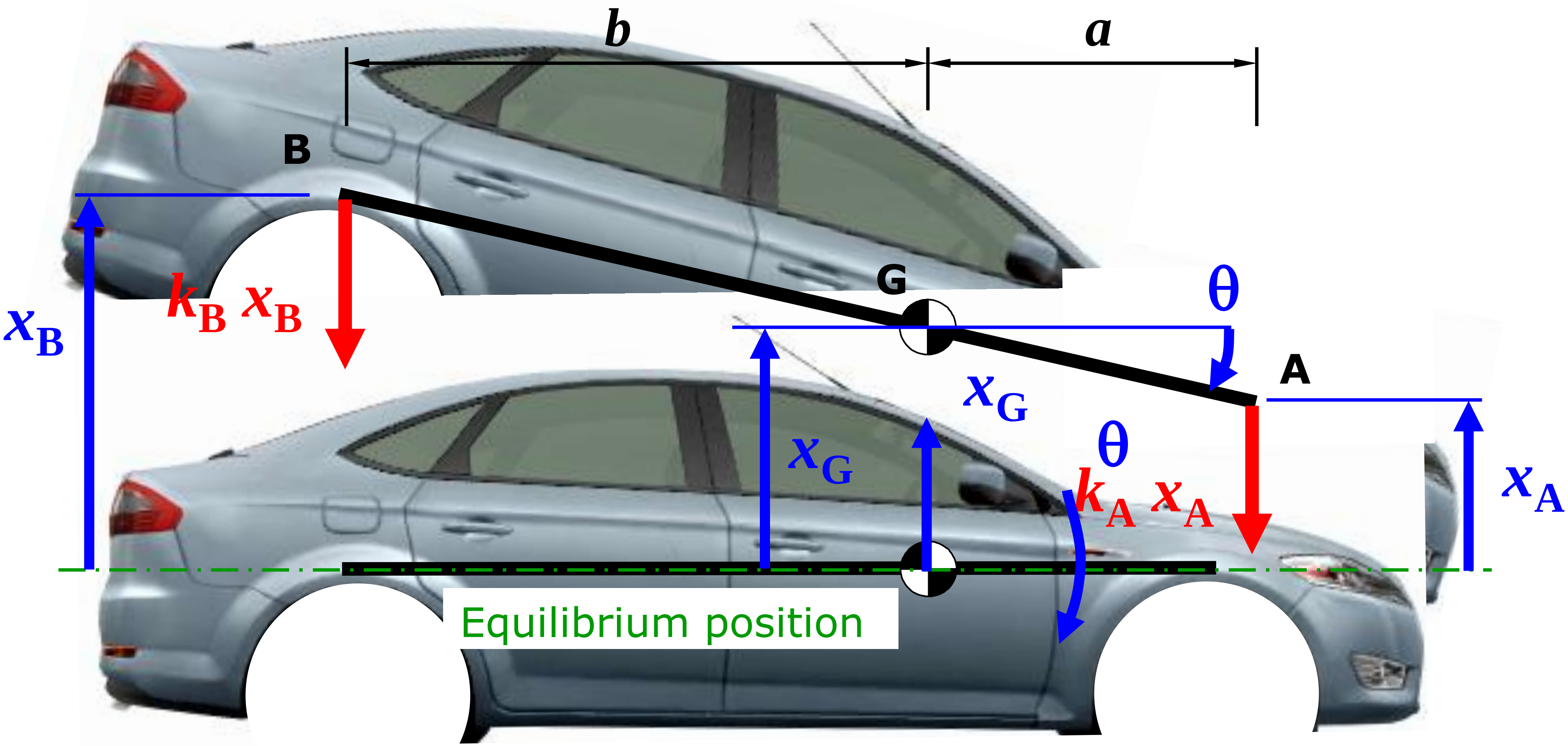
A **NO** - provided the equations of motion are right

Q *What equations of motion will we have ?*

A
1. **vertical translation**
   we **will** need the absolute acceleration of the centre of mass
2. **angular motion about G** (there is no fixed axis on AB)
   we **will** need the angular acceleration of AB

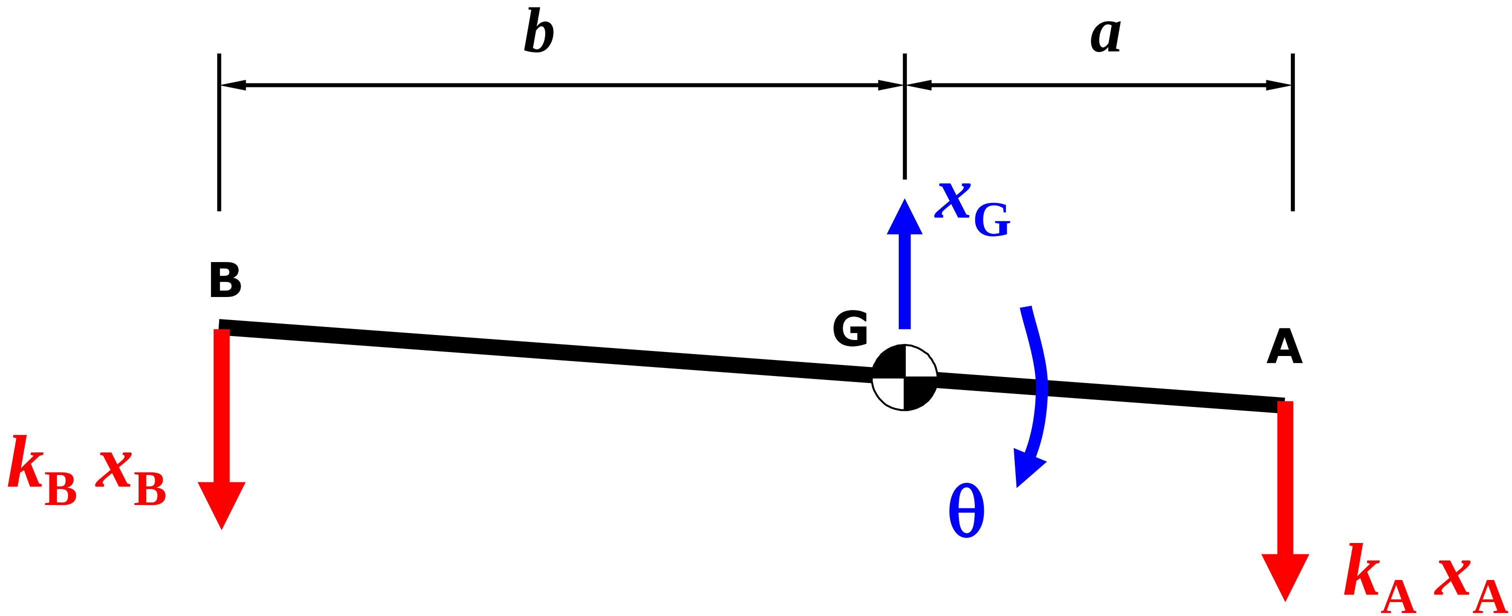
This suggests x_G and θ would be good choices

STEP 2 Free Body Diagram



For small θ, increases in spring lengths are

$$\left.\begin{aligned} x_A &= x_G - a\theta \\ x_B &= x_G + b\theta \end{aligned}\right\} \quad (1)$$



STEP 3 Equations of motion

$\uparrow x_G$

$$-k_A\left(x_G - a\theta\right) - k_B\left(x_G + b\theta\right) = m\,\ddot{x}_G$$

G $\curvearrowright \theta$

$$+k_A\left(x_G - a\theta\right).a - k_B\left(x_G + b\theta\right).b = I_G\,\ddot{\theta}_G$$

$$-k_A\left(x_G - a\theta\right) - k_B\left(x_G + b\theta\right) = m\,\ddot{x}_G$$

$$+k_A\left(x_G - a\theta\right).a - k_B\left(x_G + b\theta\right).b = I_G\,\ddot{\theta}_G$$

Re-arrange

$$m\,\ddot{x}_G + \left(k_A + k_B\right)x_G + \left(b\,k_B - a\,k_A\right)\theta = 0$$

$$I_G\,\ddot{\theta} + \left(b\,k_B - a\,k_A\right)x_G + \left(a^2 k_A + b^2 k_B\right)\theta = 0$$

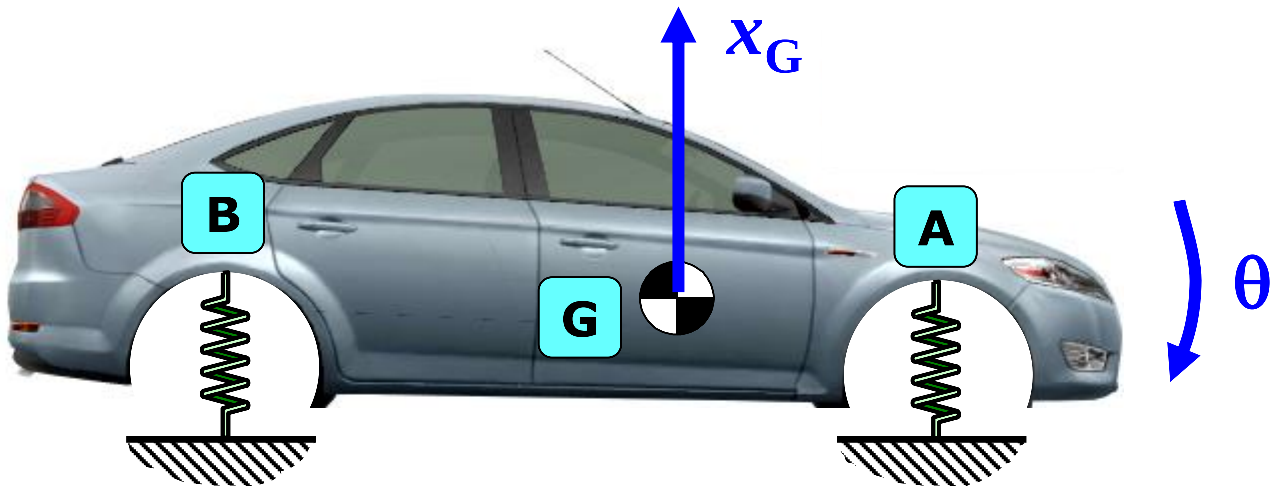
In matrix form

$$\begin{bmatrix} m & 0 \\ 0 & I_G \end{bmatrix}\begin{Bmatrix} \ddot{x}_G \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_A + k_B & b\,k_B - a\,k_A \\ b\,k_B - a\,k_A & a^2 k_A + b^2 k_B \end{bmatrix}\begin{Bmatrix} x_G \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Check leading diagonals & symmetry

Example 2 **2D vehicle model**

(coupled bounce and pitch)



Equations of motion in matrix form

$$\begin{bmatrix} m & 0 \\ 0 & I_G \end{bmatrix} \begin{Bmatrix} \ddot{x}_G \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_A + k_B & b\,k_B - a\,k_A \\ b\,k_B - a\,k_A & a^2 k_A + b^2 k_B \end{bmatrix} \begin{Bmatrix} x_G \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Natural Frequencies and Mode Shapes

For a solution, **EITHER** put $x_G(t) = X_G \cos\omega t$ and $\theta(t) = \Theta \cos\omega t$

OR write $[Z] = [K] - \omega^2 [M]$ to give

$$[Z]\begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Both methods lead to

$$\begin{bmatrix} k_A + k_B - m\omega^2 & b\,k_B - a\,k_A \\ b\,k_B - a\,k_A & a^2 k_A + b^2 k_B - I_G\,\omega^2 \end{bmatrix} \begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} (2a) \\ (2b) \end{matrix}$$

To find the **Natural Frequencies**, solve $\det[Z] = 0$

To find the **Mode Shapes**, take (2a) **or** (2b)

$$\begin{bmatrix} k_A + k_B - m\omega^2 & b\,k_B - a\,k_A \\ b\,k_B - a\,k_A & a^2 k_A + b^2 k_B - I_G\,\omega^2 \end{bmatrix} \begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{matrix} (2a) \\ (2b) \end{matrix}$$

(2a) is $$\left(k_A + k_B - m\omega^2\right) X_G + \left(b\,k_B - a\,k_A\right)\Theta = 0$$

Let Θ = 1 rad

Hence $$\begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} \left(a\,k_A - b\,k_B\right) / \left(k_A + k_B - m\omega^2\right) \\ 1.0 \end{Bmatrix} \quad (3)$$

Substituting each natural frequency into (3) will give the corresponding mode shape

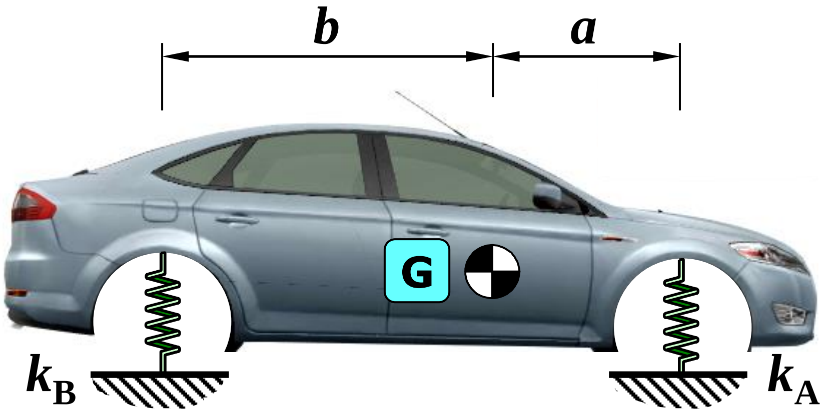
Numerical Example

DATA

m = 900 kg $\qquad I_G$ = 1000 kgm^2

k_A = 25 kN/m $\qquad k_B$ = 10 kN/m

a = 1 m $\qquad b$ = 2 m



Natural Frequencies

Substituting into the Frequency Equation $\qquad \det\left[Z\right] = 0$

$$\begin{vmatrix} 35\times10^3 - 900\omega^2 & -5\times10^3 \\ -5\times10^3 & 25\times10^3 + 40\times10^3 - 1000\omega^2 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} 35 - 0.9\omega^2 & -5 \\ -5 & 65 - \omega^2 \end{vmatrix} = 0$$

Expanding gives $$0.9\,\omega^4 - 93.5\,\omega^2 + 2250 = 0$$

Roots $\omega_{n1}^2 = 37.8\ \text{s}^{-2}$ and $\omega_{n2}^2 = 66.0\ \text{s}^{-2}$

Hence, $\omega_{n1} = \mathbf{0.98\ Hz}$ and $\omega_{n2} = \mathbf{1.29\ Hz}$

Mode Shapes

$$\begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} (a\,k_A - b\,k_B)/(k_A + k_B - m\omega^2) \\ 1.0 \end{Bmatrix} \qquad (3)$$

Mode #1

Substituting for ω_{n1}^2

$$\begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 5.10 \\ 1.0 \end{Bmatrix}$$

To visualise the mode shape, $\begin{Bmatrix} X_A \\ X_B \end{Bmatrix}$ is more convenient

From (1)

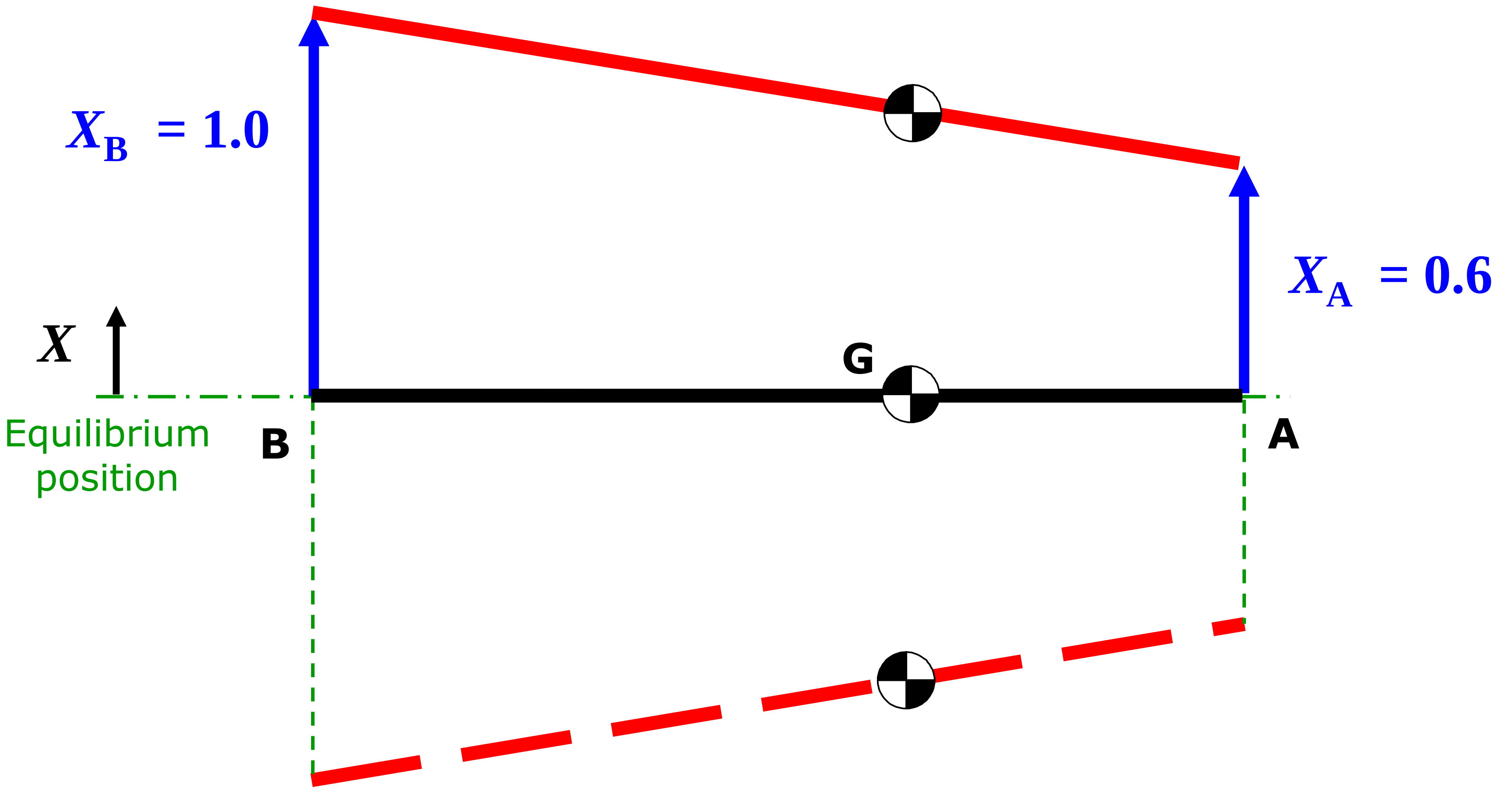
$$\begin{Bmatrix} X_A \\ X_B \end{Bmatrix} = \begin{Bmatrix} X_G - a\,\Theta \\ X_G + b\,\Theta \end{Bmatrix} = \begin{Bmatrix} 5.10 - 1\times 1 \\ 5.10 + 2\times 1 \end{Bmatrix} = \begin{Bmatrix} 4.10 \\ 7.10 \end{Bmatrix}$$

Normalising

$$\begin{Bmatrix} X_A \\ X_B \end{Bmatrix} = \begin{Bmatrix} 0.58 \\ 1.0 \end{Bmatrix}$$

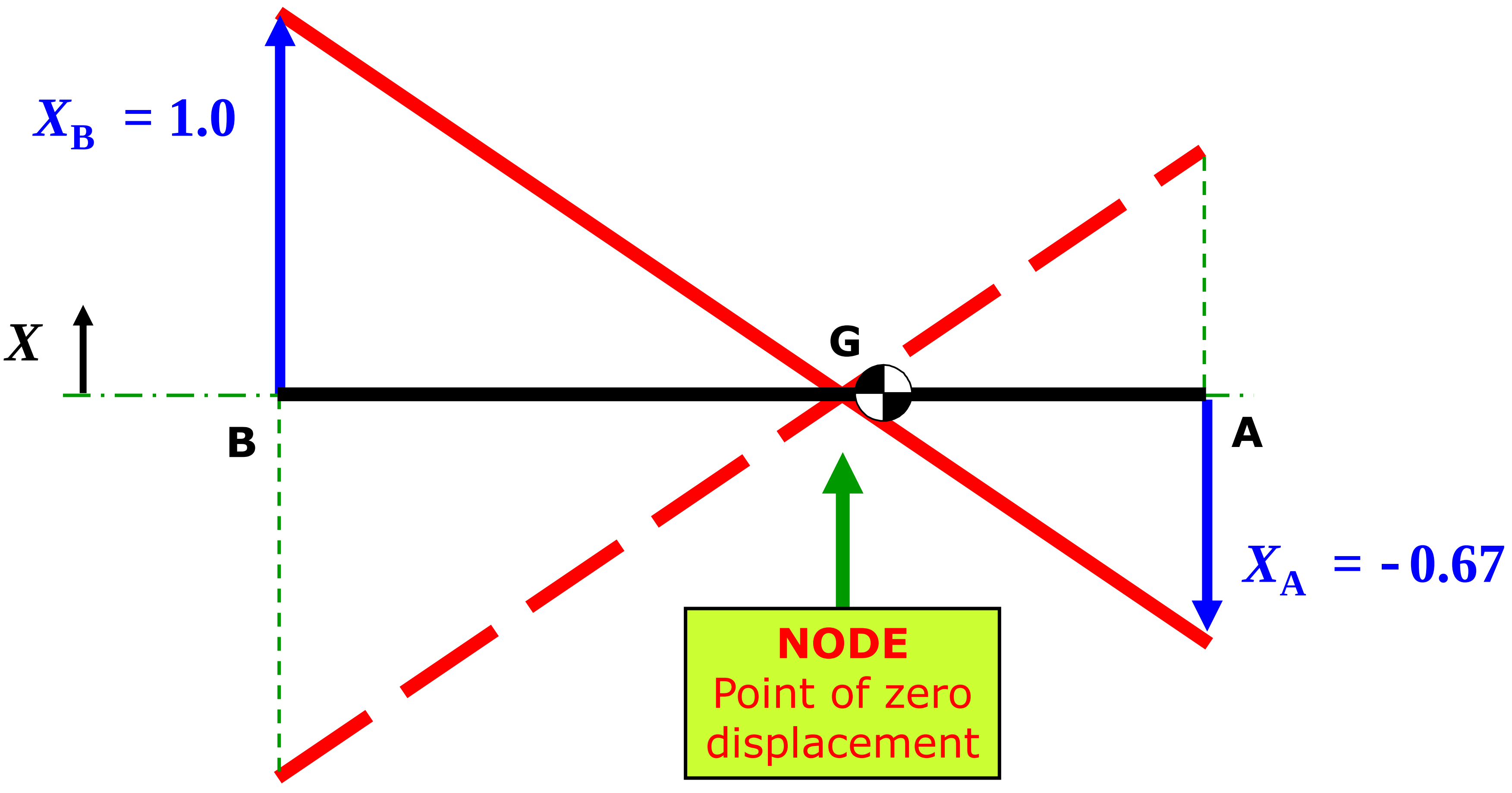
Mode #1

$$\begin{Bmatrix} X_A \\ X_B \end{Bmatrix} = \begin{Bmatrix} 0.58 \\ 1.0 \end{Bmatrix}$$

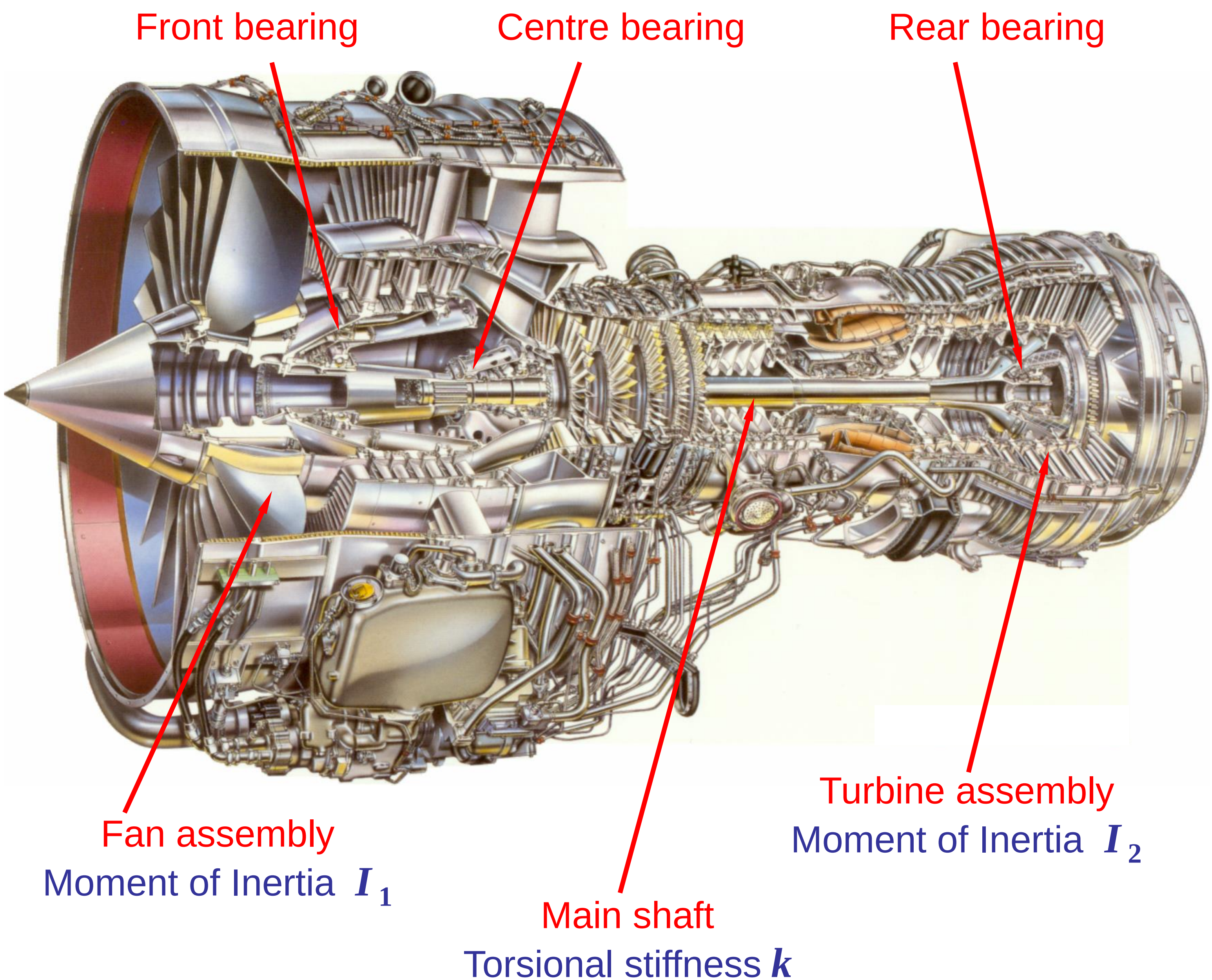


Mode #2

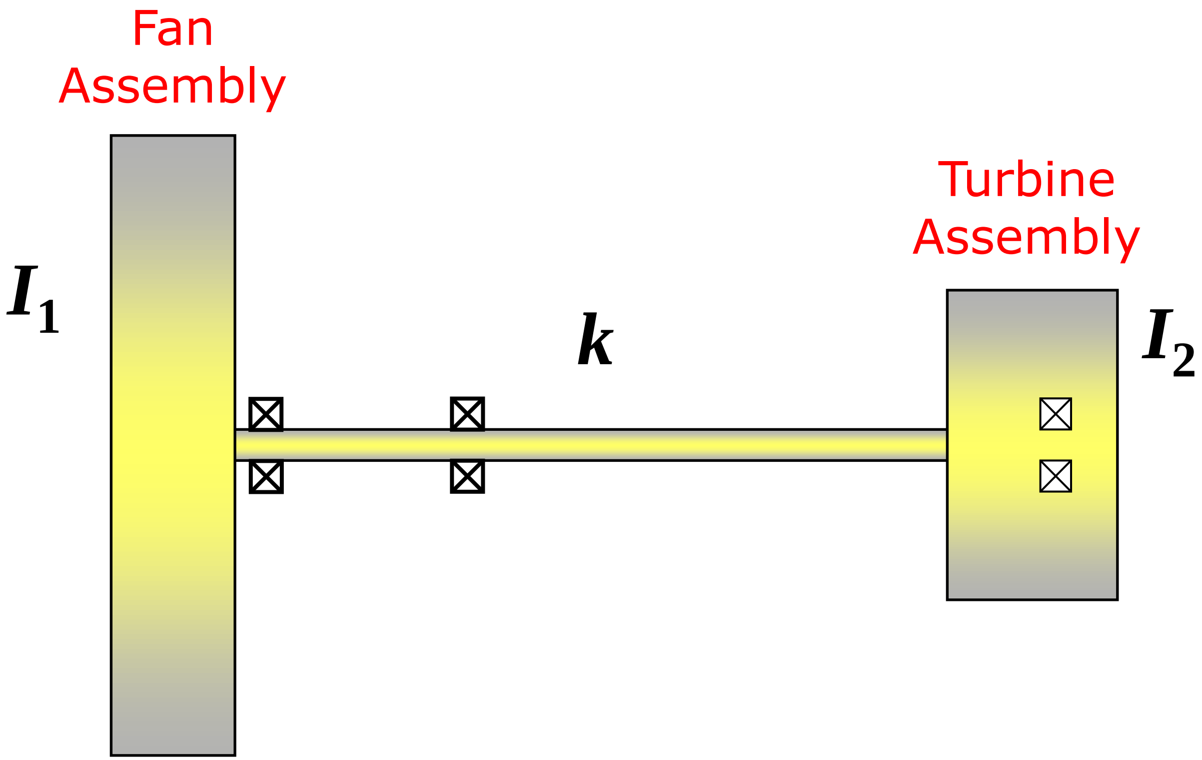
$$\begin{Bmatrix} X_G \\ \Theta \end{Bmatrix} = \begin{Bmatrix} -0.205 \\ 1.0 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} X_A \\ X_B \end{Bmatrix} = \begin{Bmatrix} -0.67 \\ 1.0 \end{Bmatrix}$$



Torsional Systems Main Drive Shaft of the V2500 Jet Engine



STEP 1 Dynamic mass-spring model



STEP 2 Free body diagrams

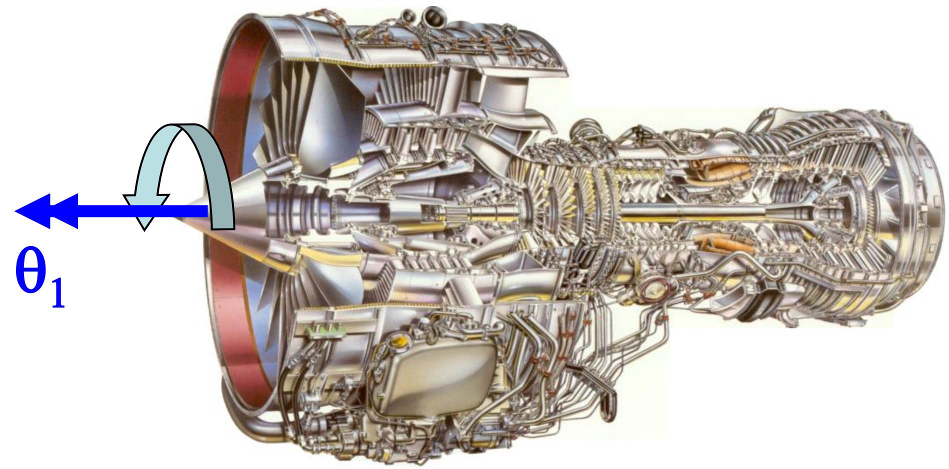
(i) Remove shaft

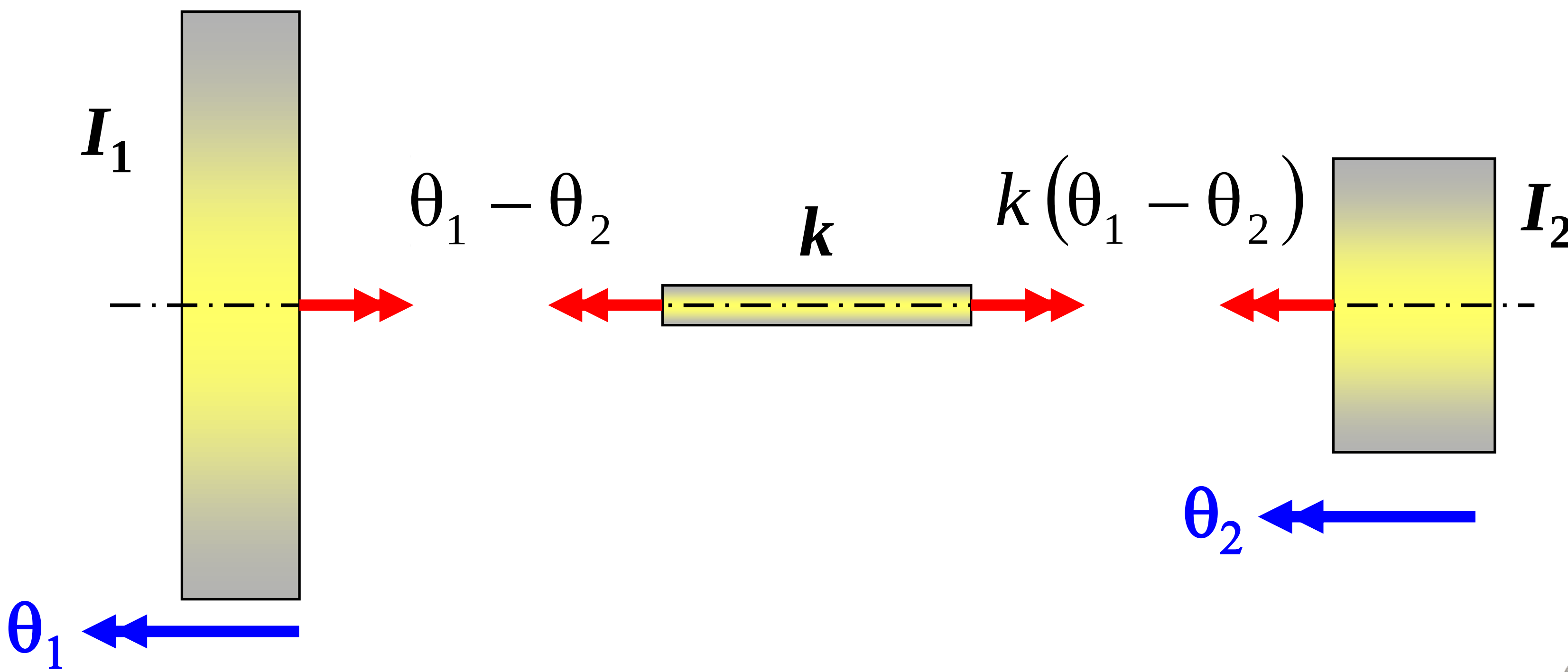
(ii) Add rotational coordinates

(iii) Add torque reactions

Twist in the shaft?

Torque direction?





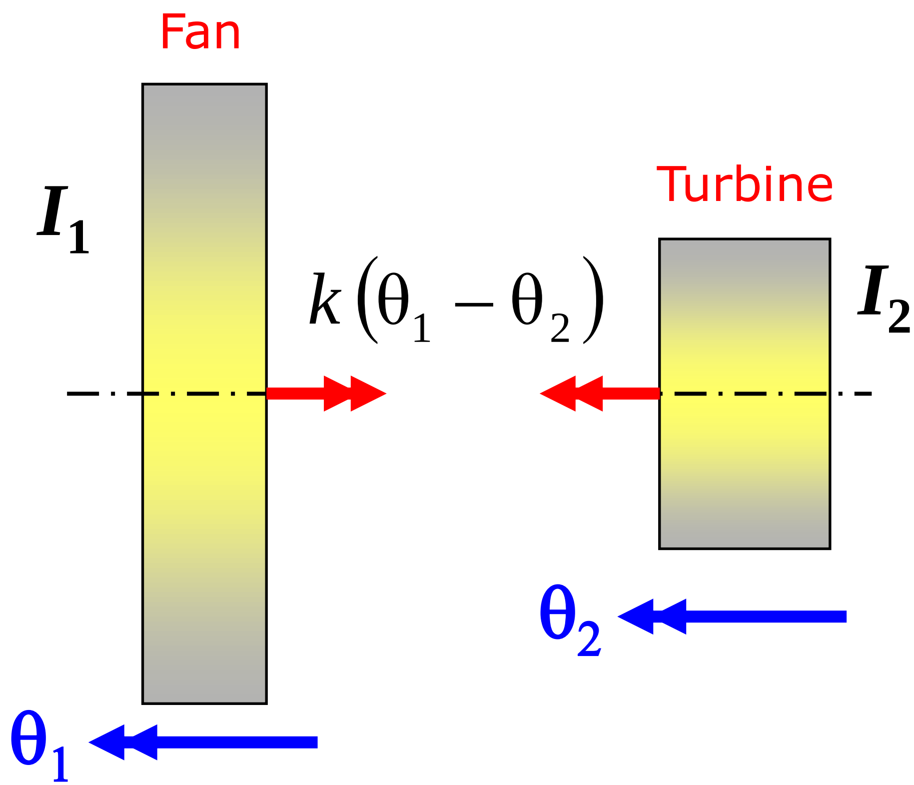
STEP 3 Equations of motion

Fan

θ_1 $\quad -k\left(\theta_1-\theta_2\right)$

Turbine

θ_2 $\quad +k\left(\theta_1-\theta_2\right)$



$$I_1\ddot{\theta}_1 + k\theta_1 - k\theta_2 = 0$$

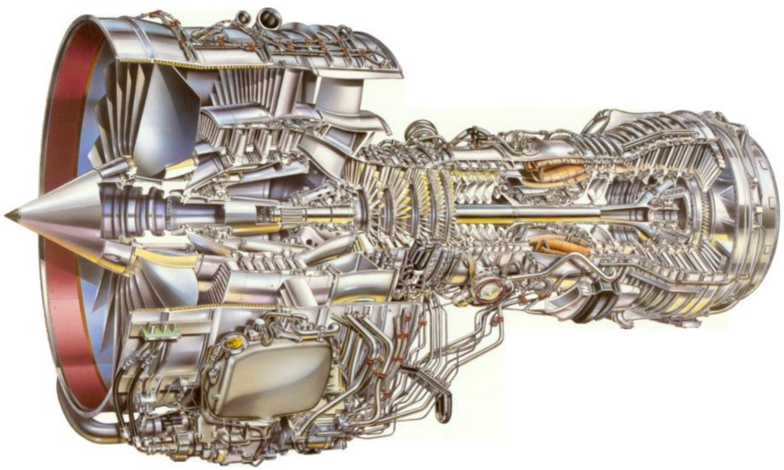
or

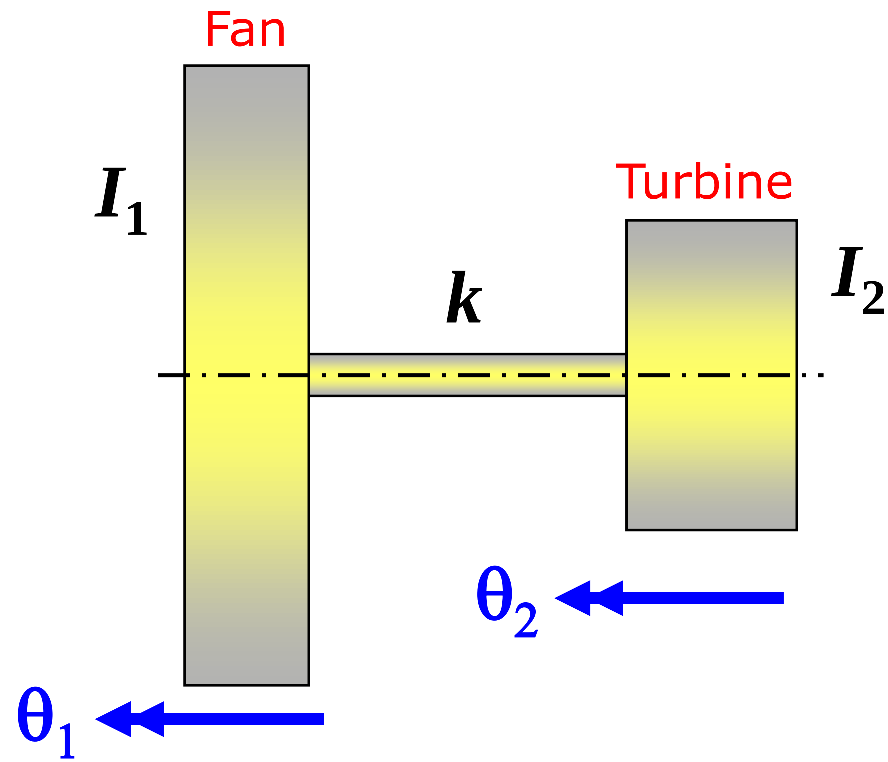
$$I_2\ddot{\theta}_2 - k\theta_1 + k\theta_2 = 0$$

or

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Check leading diagonals & symmetry





Equations of motion

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substitute $\theta_1(t) = \Theta_1 \cos\omega t$ and $\theta_2(t) = \Theta_2 \cos\omega t$

$$\begin{bmatrix} k - I_1\omega^2 & -k \\ -k & k - I_2\omega^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} (1a) \\ (1b) \end{matrix}$$

or

$$[Z]\{\Theta\} = \{0\}$$

To find the **Natural Frequencies**, solve $\det[Z] = 0$

The **Frequency Equation** for this example is

$$I_1 I_2 \omega^4 - k(I_1 + I_2)\omega^2 = 0$$

Roots $\omega_{n1}^2 = 0$ and $\omega_{n2}^2 = \dfrac{k(I_1 + I_2)}{I_1 I_2}$

To find the **Mode Shapes**, substitute roots into (1a) or (1b)

$$\begin{bmatrix} k - I_1\omega^2 & -k \\ -k & k - I_2\omega^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} (1a) \\ (1b) \end{matrix}$$

Using (1b) and putting $\Theta_2 = 1$

$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} \left(k - I_2\omega^2\right)/k \\ 1.0 \end{Bmatrix} \qquad (2)$$

Mode #1 has $\omega_{n1} = 0$ **!?**

From (2) $\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ or $\Theta_1 = \Theta_2$

Q What is implied by vibration at zero frequency?

A Continuous rotation occurs with **NO TWISTING** of the shaft

That is $\theta_1(t) = \theta_2(t) = \Omega t$ where Ω is the shaft speed

The zero frequency, combined with the mode shape, describe a **rigid body mode**

ANY structure that is capable of moving without deformation (this is true of any structure not connected to ground) **WILL** have one (or more) **rigid body modes** with $\omega_n = 0$.

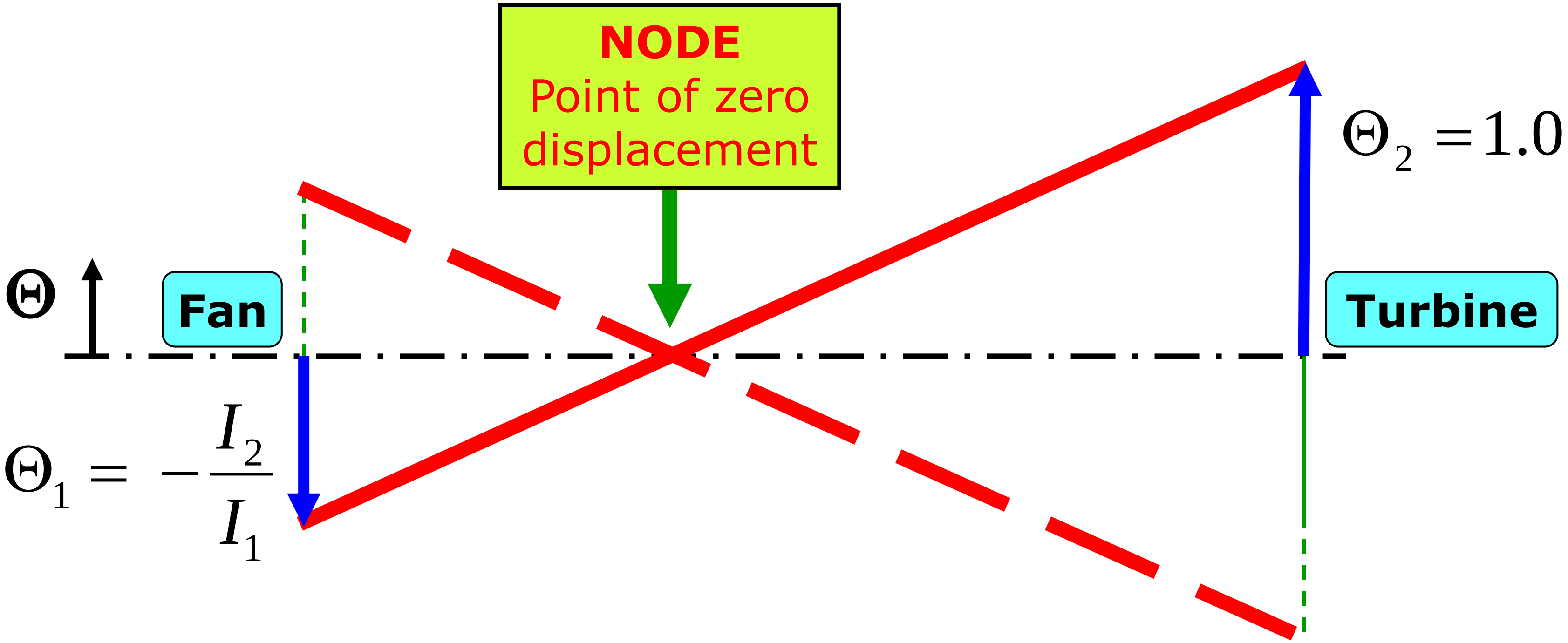
It follows that **the frequency equation will not contain a constant term**. Since you can tell in advance that this should be the case, it's a useful check that the frequency equation is correct.

$$I_1 I_2 \omega^4 - k\left(I_1 + I_2\right)\omega^2 = 0$$

Mode #2

$$\omega_{n2} = \sqrt{\frac{k\left(I_1 + I_2\right)}{I_1 I_2}}$$

From (2) $\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} -I_2/I_1 \\ 1.0 \end{Bmatrix}$



This torsional vibration is superimposed on the continuous rotation of the shaft

NEXT