7 Asymmetrical Bending

Learning Summary

By the end of this section you should have learnt,

1. Know that an asymmetric cross-section, in addition to its 2nd moments of area about the x and y axes, I_x and I_y, possesses a geometric quantity called the Product Moment of Area, I_{xy}, with respect to these axes (knowledge);
2. Be able to calculate the 2nd moments of area and the product moment of area about a convenient set of axes (application);
3. Know that an asymmetric section will have a set of axes at some orientation for which the product moment of area is zero and that these axes are called the Principal Axes (knowledge);
4. Know that the 2nd moments of area about the principal axes are called the principal 2nd moments of area (knowledge);
5. Be able how to determine the 2nd moments of area and the product moment of area about any oriented set of axes, including the principal axes, using a Mohr's circle construction (application);
6. Understand that it is convenient to analyses the bending of a beam with an asymmetric section by resolving bending moments onto the principal axes of the section (knowledge);
7. Be able to follow a basic procedure for analysing the bending of a beam with an asymmetric cross-section (application).

7.1 Introduction

The beam bending equation, $\frac{M}{I}=\frac{\sigma}{y}=\frac{E}{R}$, has been derived and is generally used to determine stresses in a beam with a symmetrical cross-section. The symmetry is usually about an axis perpendicular to the neutral axis of the section. For a section where this symmetry does not apply, i.e. asymmetric sections, a complication arises, making bending analysis more difficult. In these cases, applying a bending moment will, in general, result not only in bending about that axis but also in simultaneous bending about the perpendicular axis i.e. there is an interaction effect. To analyse such sections we introduce a new geometric quantity called the Product Moment of Area and this leads to the concept of Principal 2nd Moments of area and Principal Axes for the section. These are axes for which the Product Moment of area is zero and the above interaction effect during bending does not occur. Thus, it is convenient to analyse the bending of asymmetric sections about these axes. In this section, we will look at the theory behind this effect and develop a general procedure for dealing with asymmetrical bending situations.

7.2 Second moments of area of a complex shaped cross-section

7.2.1 2nd Moments of Area about Parallel Axes

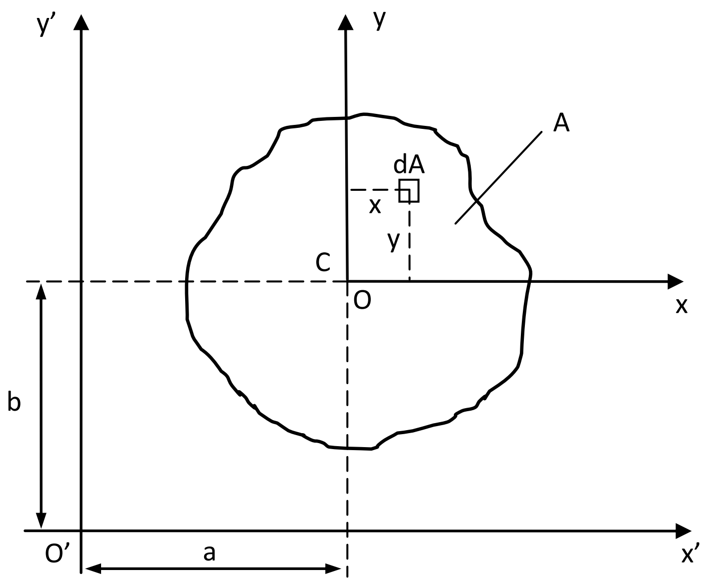


Figure 7.1. Arbitrarily shaped cross-section

Consider an arbitrary shaped cross-section, as shown in Figure 7.1. The centroid of the area, C, is at the origin, O, of the O-x-y axes set. A parallel axes set, O'-x'-y', also exists, distance a and b from the O-x-y axes set, as shown in the figure. The centroid of area, C, is positioned at co-ordinates $(x',y') = (a,b)$ in this parallel axes set.

We know that the 2nd moments of area, I_x and I_y, of the section with respect to the x and y axes are given by,

$$I_x = \int_A y^2 dA \quad \text{and} \quad I_y = \int_A x^2 dA$$

i.e. the product of an element of area, dA, and its distance squared from the particular axis (x or y), integrated over the full cross-sectional area, A.

The **Parallel Axis Theorem** allows the calculation of the 2nd moments of area, $I_{x'}$ and $I_{y'}$, with respect to the x' and y' axes as follows,

$$I_{x'} = I_x + Ab^2 \qquad \textbf{[1]}$$

and

$$I_{y'} = I_y + Aa^2 \qquad \textbf{[2]}$$

I_x and I_y are the 2nd moments of area about a set of axes through the centroid and are always the minimum 2nd moments. $I_{x'}$ and $I_{y'}$ will always be greater because the second terms in equations [1] and [2] are always positive as the distances between the axes, a and b, are squared.

7.2.2 The Product Moment of Area

We now introduce a new quantity, the **product moment of area**, I_{xy}, which is defined as,

$$I_{xy} = \int_A xy \, dA$$

I_{xy} is the summation of the elements of area multiplied by the product of their co-ordinates. We can now develop the parallel axis theorem for the product moment of area as follows,

$$I_{x'y'} = \int_A x'y'\,dA = \int_A (x+a)(y+b)\,dA$$
$$= \int_A xy\,dA + a\int_A y\,dA + b\int_A x\,dA + ab\int_A dA$$

but, $\int_A y\,dA$ and $\int_A x\,dA$ are both zero because the origin of axes Oxy is at the centroid of area, C . Thus,

$$I_{x'y'} = I_{xy} + abA \qquad \textbf{[3]}$$

This is the **Product Parallel Axis Theorem**. Again, I_{xy} is the product moment of area about a set of axes through the centroid. In this case, $I_{x'y'}$, can be either positive or negative, depending on the signs of a and b.

7.2.3 Principal 2nd Moments of Area

Equations [1], [2] and [3] can be used to plot a Mohr's circle as shown in Figure 7.2. 2nd moments are plotted on the x-axis and the product moments are plotted on the y-axis [*note that the y-axis for the circle is positive upwards, unlike Mohr's circle for stress which is positive downwards for shear*].

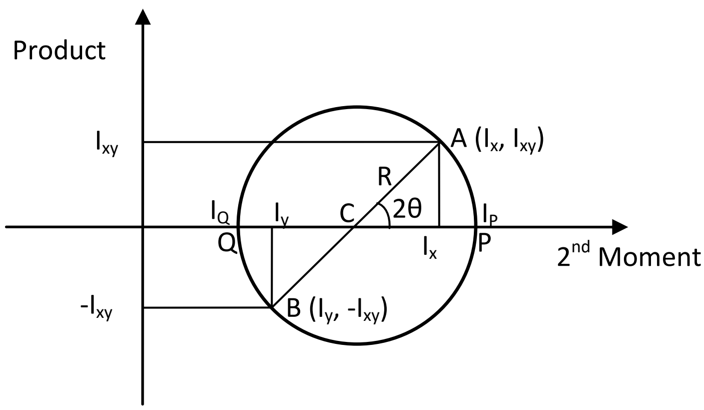


Figure 7.2. Mohr's Circle

Point A on the circle has co-ordinates which correspond to the first 2nd moment and the product moment, i.e. (I_x, I_{xy}). Point B on the circle has co-ordinates which correspond to the second 2nd moment and the negative product moment, i.e. (I_y, I_{yx}=-I_{xy}). These two points enable the circle to be drawn.

The centre of the circle, C, and radius, R, are given by,

$$Centre\ C = \frac{I_x + I_y}{2} \qquad [4]$$

and

$$Radius\ R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^{\ 2}} \qquad [5]$$

The points P and Q on the circle correspond to the **Principal Planes** for which the product moment of areas are zero and the 2nd moments are the **Principal 2nd Moments of Area**, I_P and I_Q. Their magnitudes are given by,

I_P = Centre + Radius

and

I_Q = Centre – Radius

where the centre and radius are given by equations [4] and [5].

Thus knowing I_x, I_y and I_{xy}, The principal 2nd moments of area, I_P and I_Q, can be determined.

The angle of the principal axes with respect to the x-y axes is the angle θ, where 2θ is shown in Figure 7.2 and is given by,

$$\sin 2\theta = \frac{I_{xy}}{R}$$

or alternatively $tan\,2\theta = \dfrac{I_{xy}}{\left(\dfrac{I_x - I_y}{2}\right)}$

7.3 Symmetric Sections

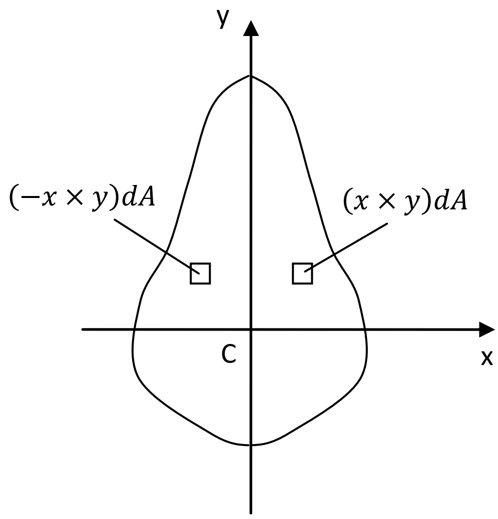


Figure 7.3. Symmetric section

Figure 7.3 shows a section where one axis (the y-axis in this case) is an axis of symmetry. The sum of the contributions to the product moment of area from elements of area, dA, on opposite sides of the axis of symmetry will cancel out because of the change of sign of the x co-ordinate. Thus, in general, if a section has an axis of symmetry, then I_{xy} is zero.

7.4 Key points about the Mohr's circle for 2nd moments of area

1. The +ve upward direction for the product moment ensures that rotation in the Mohr's circle has the same sense as the rotation of the axes in space.
2. I_P and I_Q are both +ve.
3. If $I_P = I_Q$, all product moments are zero and all axes in all directions are principal axes e.g. this is the case for a circular section.

4. The sign of the product moment is important. $I_{xy} = \int_A xy\, dA$ is associated with the x-axis and can be +ve or –ve. The product moment associated with the y-axis is $I_{yx} = -I_{xy}$.

7.5 Summary of procedure to calculate the Principal 2nd Moments of Area and the directions of the Principal Axes

1. Divide the cross-section into subsections for which their centroid of areas and 2nd moments of area about their own axes can be determined.
2. Choose a convenient set of orthogonal axes with its origin at the centroid of the full cross-section.
3. Use the parallel axis theorem to determine the 2nd moments of area and the product moment of area for the full cross-section.
4. Use a Mohr's circle construction to determine the Principal 2nd Moments of Area and the directions of the Principal Axes.

7.6 Worked Example – Principal 2nd Moments of Area

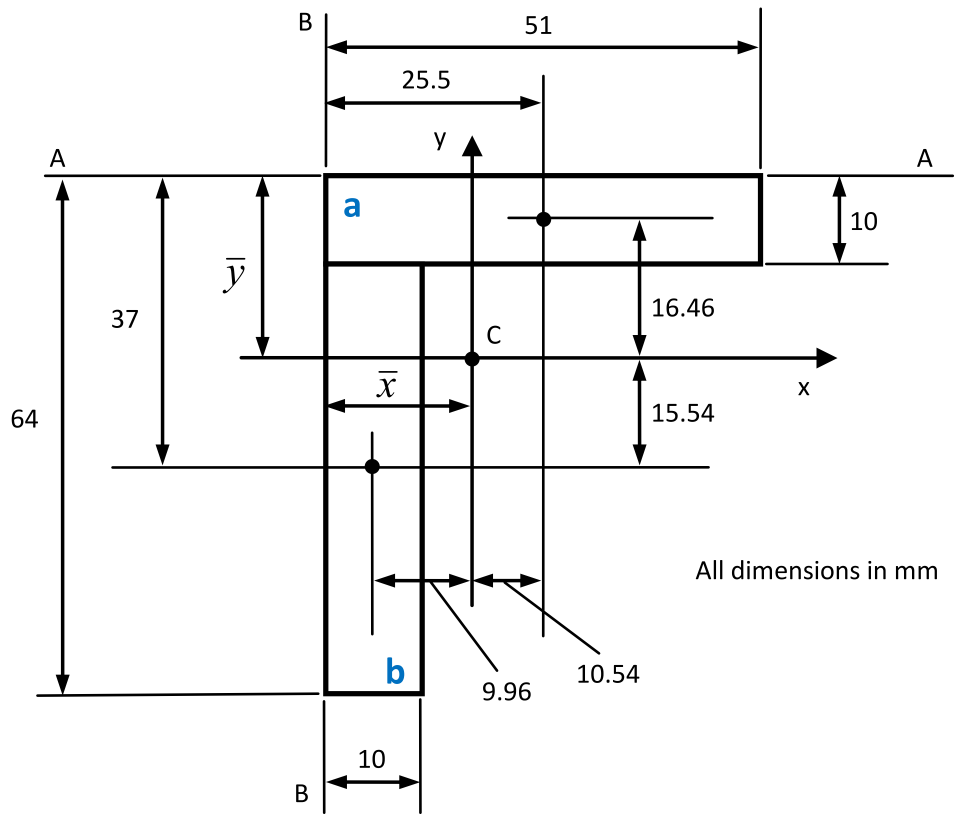


Figure 7.4. Worked example cross-section

Figure 7.4 shows an asymmetric angle cross-section. Determine:

(a) the Principal 2nd Moments of Area

(b) the directions of the Principal Axes

The section is divided into two rectangular subsections 1 and 2.

Position of the Centroid:

Total Area = 51x10 + 54x10 = 1050 mm^2

Taking moments of the areas about the datum AA,

$$1050 \times \bar{y} = (51 \times 10) \times 5 + (54 \times 10) \times 37$$
$$\therefore \bar{y} = 21.46\, mm$$

Taking moments of the areas about the datum BB,

$$1050 \times \bar{x} = (51 \times 10) \times 25.5 + (54 \times 10) \times 5$$
$$\therefore \bar{x} = 14.96\, mm$$

2nd Moments of Area about a convenient set of axes:

The x and y axes are drawn as a convenient set of axes through the centroid.

Using the parallel axis theorem,

$$I_{x'} = \left(\frac{51 \times 10^3}{12} + 51 \times 10 \times 16.46^2 \right) + \left(\frac{10 \times 54^3}{12} + 10 \times 54 \times -15.54^2 \right)$$
$$= \underline{404{,}051\ mm^4}$$

$$I_{y'} = \left(\frac{10 \times 51^3}{12} + 10 \times 51 \times 10.54^2 \right) + \left(\frac{54 \times 10^3}{12} + 54 \times 10 \times -9.96^2 \right)$$
$$= \underline{225{,}268\ mm^4}$$

And the product parallel axis theorem,

$$I_{x'y'} = (0 + 51 \times 10 \times 10.54 \times 16.46) + (0 + 54 \times 10 \times (-9.96) \times (-15.54))$$
$$= \underline{172{,}059\ mm^4}$$

Note that, in the product moment of area calculation above, the product moment of each subsection about its own axis is zero due to the symmetry of each subsection. It is also important that the correct sign for the co-ordinates of each subsection centroid with respect to the full cross-section centroid are taken. Thus, for subsection 1, the co-ordinates are both positive (10.54 and 16.46), while for subsection 2, they are both negative (-9.96 and -15.54).

Mohr's Circle:

A Mohr's circle can now be drawn to represent the axes about which $I_{x'}$, $I_{y'}$ and $I_{x'y'}$ act, as shown in Figure 7.5. The centre and radius are calculated as follows,

$$Centre\ C = \frac{I_{x'} + I_{y'}}{2} = 314{,}659\ mm^4$$

$$Radius\ R = \sqrt{\left(\frac{I_{x'} - I_{y'}}{2}\right)^2 + I_{x'y'}{}^2} = 193{,}895\ mm^4$$

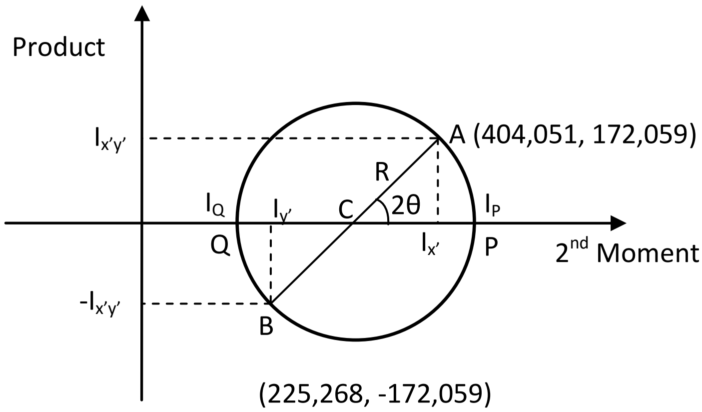


Figure 7.5. Worked example Mohr's circle

Principal 2nd Moments of area:

The principal 2nd moments of area can now be determined from the circle as follows,

I_P = C + R = 508, 554 mm^4

I_Q = C - R = 120, 764 mm^4

and the angle, θ, of the principal axes with respect to the x-axis is given by,

$$sin\,2\theta = \frac{I_{x'y'}}{R} = \frac{172{,}059}{193{,}895} = 0.887$$

$$\therefore \underline{\theta = 31.27°}$$

From the Mohr's circle it can be seen that the principal axis 1 i.e. the p-axis is 31.27° clockwise from the x-axis. The principal axes can now be drawn on a sketch of the element as shown in Figure 7.6.

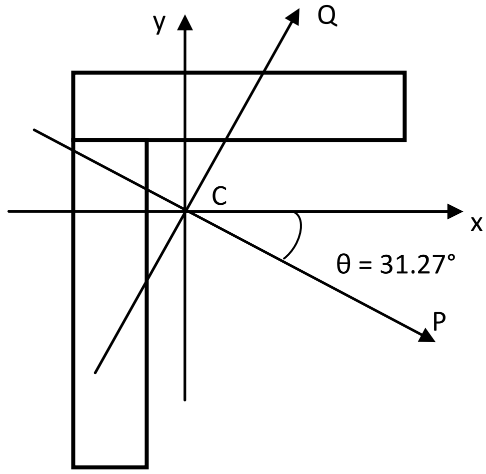


Figure 7.6. Worked example schematic solution

7.7 Bending of beams with asymmetric sections

Figure 7.7 shows an arbitrary cross-section of a beam subjected to a bending moment, M, acting at an angle θ to the x-axis. The origin of the x-y axes coincides with the centroid of the section. The bending moment has two components, M_x and M_y, as shown, acting about the x-axis and y-axis respectively. [*note that the bending moment and its components are drawn in vector form with a double arrow head. The right hand screw rule defines the sense of each bending moment component as shown in the figure]*

Assume that bending takes place only about the x-axis i.e. O-x is the neutral axis. Then, the bending stress, σ, is proportional to the distance, y, from the neutral axis, or alternatively,

$$\sigma = c.y$$

where c is an arbitrary constant

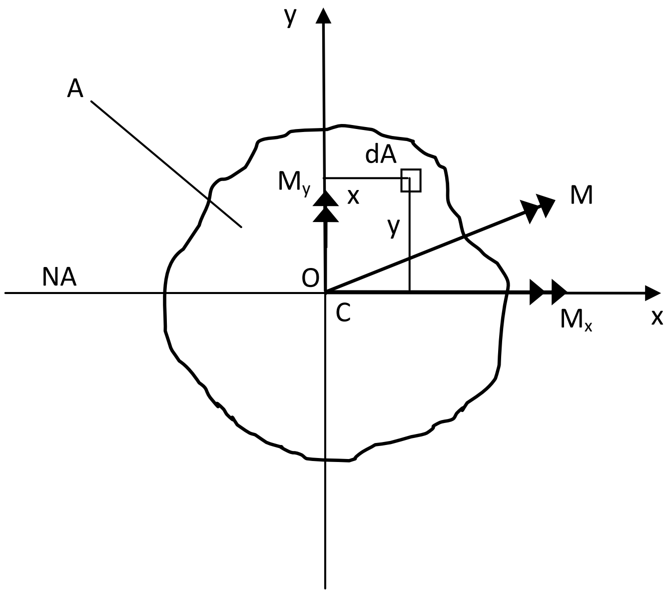


Figure 7.7. Arbitrary cross-section of a beam subjected to a bending moment, M

The resultant moment about the x-axis is given by the sum of moments of the forces acting on each elemental area in the cross-section. In the limit, this sum can be written as an integral as follows,

$$M_x = \int_A \sigma . y \, dA$$

$$= \int_A c.y.y \, dA$$

$$\therefore M_x = c.I_x \qquad \textbf{[6]}$$

where I_x = 2nd moment of area about the x-axis

But c = σ/y

$$\therefore \sigma = \frac{M_x}{I_x} y$$

which is the beam bending equation as expected.

However, there is also a resultant moment about the y-axis, as follows,

$$M_y = -\int_A \sigma.x\,dA$$

$$= -\int_A c.y.x\,dA$$

$$\therefore M_y = -c.I_{xy} \qquad \textbf{[7]}$$

where $I_{xy} = \int xy\,dA$ is the Product Moment of Area

[note the –ve sign arising because a positive stress results in a –ve moment about the y-axis]

Thus, in general, a moment has to be applied about the y-axis as well as the x-axis to produce bending about the x-axis only. A +ve moment is required about the y-axis to counterbalance the –ve moment set up by the stresses arising from M_x. This is not the case if I_{xy} is zero i.e. for sections which are symmetric about the y-axis.

To ensure bending about the x-axis only, a resultant moment $M = \sqrt{M_x^{\ 2} + M_y^{\ 2}}$ must be applied at an angle, θ, given by,

$$\theta = tan^{-1}\left(\frac{M_y}{M_x}\right)$$

$$= tan^{-1}\left(\frac{-I_{xy}}{I_x}\right)$$ (from equations [6] and [7])

The moment is only applied about the x-axis when I_{xy}=0.

Figure 7.8 illustrates the effect for a z-section. If a bending moment is applied about the x-axis only, then the stresses in the flanges will create a resulting moment about the y-axis. Consequently, bending will take place about both the x- and y- axes. This is a consequence of I_{xy} not being zero for this asymmetric section.

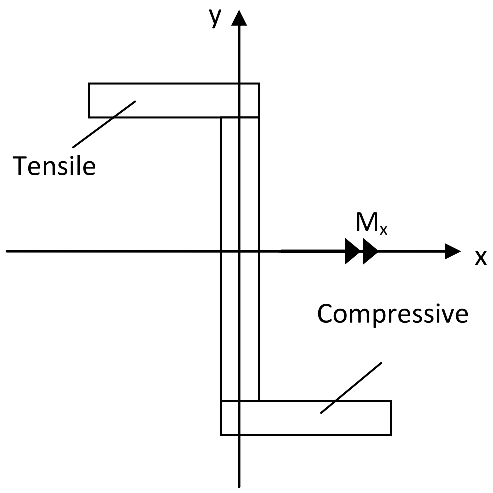


Figure 7.8. Z-section diagram

To avoid this moment coupling effect, it is usually convenient to solve bending problems by considering bending about the **Principal Axes** of a section for which the **Product Moment of Area** is zero.

7.8 Solving asymmetrical bending problems

Consider the arbitrary asymmetric section shown in Figure 7.9(a). O is the centroid and O-P and O-Q are the **Principal Axes** of the section. The principal axes are inclined at an angle θ to the x-y axes. Components of an applied moment M, i.e. M_x and M_y, act about the O-x and O-y axes respectively. Firstly, M_x and M_y are resolved onto the principal directions, as illustrated in Figure 7.9(b), giving,

$$M_P = M_x\cos\theta + M_y\sin\theta \quad \text{and} \quad M_Q = -M_x\sin\theta + M_y\cos\theta$$

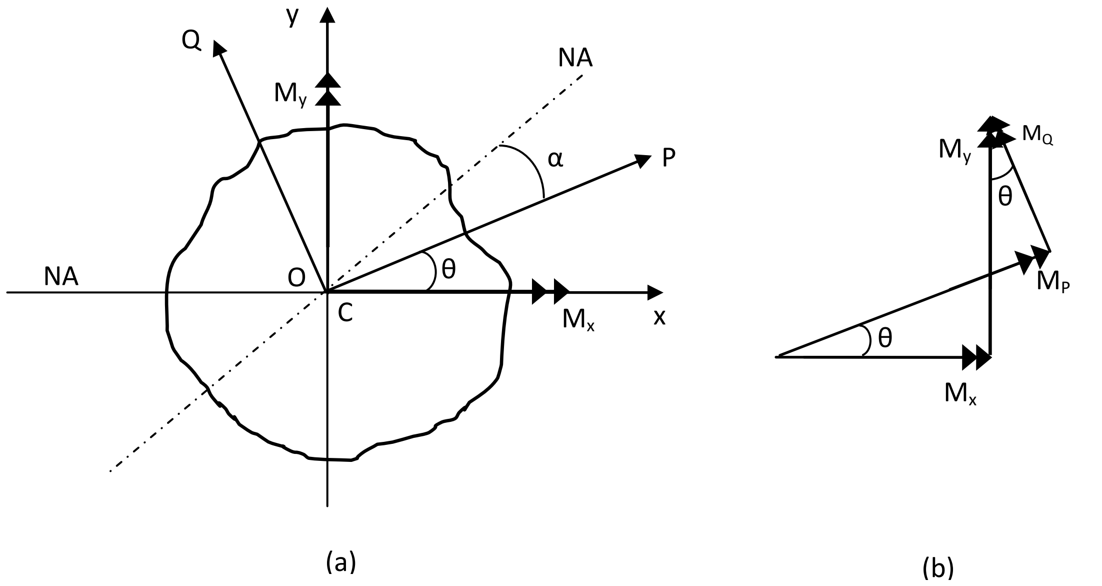


Figure 7.9. Arbitrary section and principal axes

We can now calculate the total bending stress, σ_b, at any position, (P, Q), which arises from these two bending moment components and is given by,

$$\sigma_b = \frac{M_P.Q}{I_P} - \frac{M_Q.P}{I_Q} \quad \textbf{[8]}$$

[note that when P and Q are both +ve, i.e. in the first quadrant of the P-Q axes set, a +ve M_P gives rise to a +ve bending stress while a +ve M_Q gives rise to a –ve bending stress]

The maximum stress in the section will occur at the extreme distance from the **Neutral Axis**. We therefore need to determine the position/orientation of the neutral axis which can be found by setting the bending stress, i.e. equation [8], to zero. Thus, the neutral axis occurs where,

$$\sigma_b = \frac{M_P.Q}{I_P} - \frac{M_Q.P}{I_Q} = 0$$

$$\therefore \frac{M_P.Q}{I_P} = \frac{M_Q.P}{I_Q}$$

$$\therefore \frac{Q}{P} = \frac{M_Q}{M_P}.\frac{I_P}{I_Q}$$

This value for q/p defines the angle, α, of the neutral axis, with respect to the p-axis, shown in Figure 7.9(a), as follows,

$$\alpha = tan^{-1}\left(\frac{Q}{P}\right) = tan^{-1}\left(\frac{M_Q.I_P}{M_P.I_Q}\right) \quad \textbf{[9]}$$

Equation [8] can therefore be used to determine the magnitude of the stress at any position (p,q) and equation [9] can be used to determine the orientation of the neutral axis and hence the position of the maximum stress which is at the extreme distance from the neutral axis.

7.9 Summary of the procedure for solving asymmetrical bending problems

1. Determine the Principal Axes of the section, P and Q, about which $I_{xy} = 0$.
2. Consider bending about the principal axes, i.e. resolve bending moments onto these axes.
3. Knowing M_P, M_Q, I_P and I_Q, determine the general expression for the bending stress at position (P, Q) as follows,

$$\sigma_b = \frac{M_P.Q}{I_P} - \frac{M_Q.P}{I_Q}$$

4. Determine the angle of the neutral axis with respect to the P-axis as follows,

$$\alpha = tan^{-1}\left(\frac{Q}{P}\right) = tan^{-1}\left(\frac{M_Q.I_P}{M_P.I_Q}\right)$$

5. Evaluate the bending stress at any position in the section including the extreme positions from the neutral axis which give the maximum bending stresses.

7.10 Worked Example – Asymmetrical Bending

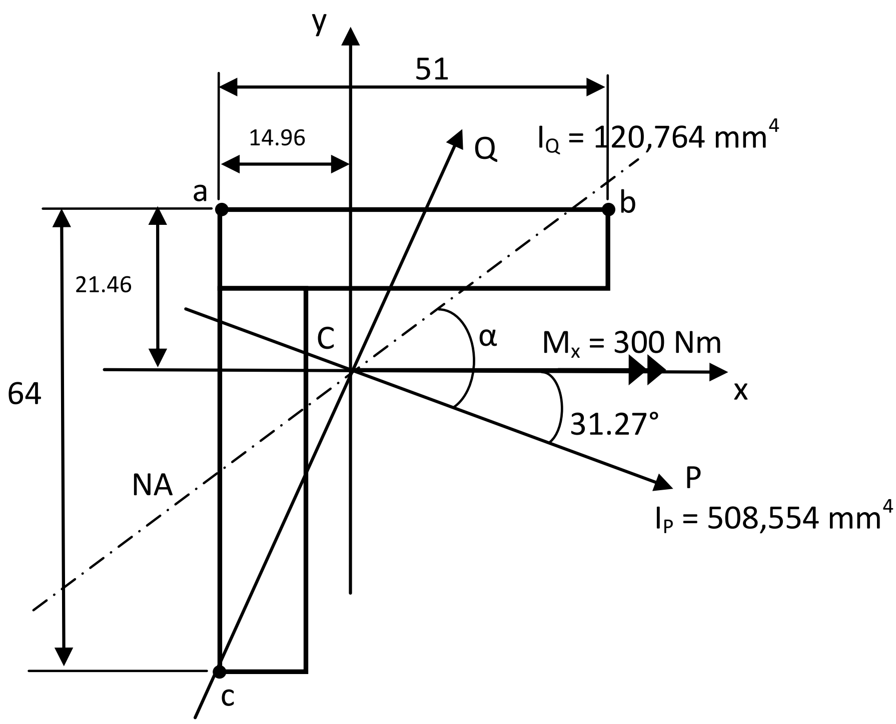


Figure 7.10. Worked example cross-section

The angle section, shown in Figure 7.10, with principal axes and principal 2nd moments of area indicated, is subjected to a bending moment of 300Nm about the x-axis. Determine:

(i) the position/orientation of the neutral axis

(ii) the bending stresses at positions a, b and c

Resolving the applied moment:

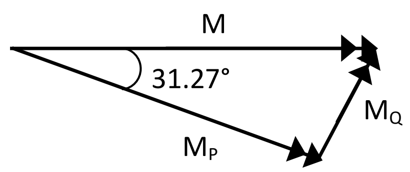


Figure 7.11. Components of applied bending

Referring to Figure 7.11(a), the components of the applied bending in the p and q directions are,

M_P = Mcos(31.27) = 0.855M

M_Q = Msin(31.27) = 0.519M

The general expression for bending stress at position (p,q) is,

$$\sigma_b = \frac{M_P.Q}{I_P} - \frac{M_Q.P}{I_Q}$$
$$= \frac{0.855 \times 300 \times 10^3 \times Q}{508{,}554} - \frac{0.519 \times 300 \times 10^3 \times P}{120{,}764}$$
$$= 0.5042Q - 1.2894P$$

Note that for P and Q in mm, this expression gives bending stress in N/mm^2 i.e. MPa.

Orientation of the neutral axis, α, with respect to the P-axis:

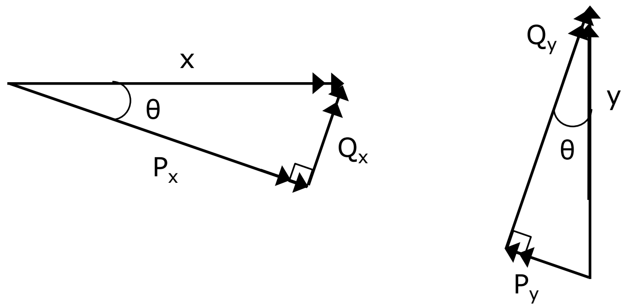
$$\alpha = tan^{-1}\left(\frac{Q}{P}\right) = tan^{-1}\left(\frac{M_Q.I_P}{M_P.I_Q}\right)$$
$$= tan^{-1}\left(\frac{1.2894}{0.5042}\right) = tan^{-1}(2.558)$$
$$= 68.64°$$

Orientation of the neutral axis with respect to the x-axis = 68.64 – 31.27 = 37.37°. These orientations are illustrated in Figure 7.10.

Bending stresses:

To determine the bending stresses at a, B and c, we need the P and Q co-ordinates of these points. Referring to Figure 7.11, the general co-ordinate transformation equations for a set of axes, P-Q, inclined at a clockwise angle, θ, from another set, x-y, are (as shown in Figure 7.12),

P = xcosθ - ysinθ and Q = xsinθ + ycosθ



$$P = P_x - P_y$$

$$Q = Q_x + Q_y$$

Figure 7.12. Transformation from axis set x-y to axis set P-Q

For this problem, the P-axis is inclined at 31.27° clockwise to the x-axis. Thus, θ = 31.27° and the above transformation equations become,

P = 0.8557x – 0.5191y and Q = 0.5191x + 0.8547y

We can now draw a table for calculating the co-ordinates of A, B and C as follows,

Position	x	y	p	q
a	-14.96	21.46	-23.92	10.88
b	36.04	21.46	19.66	37.05
c	-14.96	-42.54	9.3	-44.12

and the stresses follow from the general equation $\sigma_b = 0.5042Q - 1.2894P$, as follows,

at a: σ_a = 36.33MPa

at b: σ_b = - 6.67MPa

at c: σ_c = - 34.24MPa