Minilecture 1G: Homogeneous equations repeated roots.

Recall $y = e^{mx}$ is a solution of

$$a\, d^2y/dx^2 + b\, dy/dx + cy = 0$$

if

$$am^2 + bm + c = 0$$

Three cases
- real distinct roots — done ✓
- complex roots — done ✓
- repeated roots — now

Repeated roots

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 = 4ac$

One root: $m = m_1 = -b/2a$. Just one solution to ODE

$$y_1(x) = e^{m_1 x}$$

Need another! The other soln can be

$$y_2(x) = x e^{m_1 x}$$

$$y_2'(x) = e^{m_1 x} + m_1 x e^{m_1 x}$$

$$y_2''(x) = 2m_1 e^{m_1 x} + m_1^2 x e^{m_1 x}$$

Plug in

don't normally do these steps

$$ay_2''(x) + by_2'(x) + cy_2(x) = a(2m_1 e^{m_1 x} + m_1^2 x e^{m_1 x}) + b(e^{m_1 x} + m_1 x e^{m_1 x}) + c\, x e^{m_1 x}$$

$$= 2m_1 a e^{m_1 x} + b e^{m_1 x} + x(am_1^2 + bm_1 + c) e^{m_1 x}$$

$$= (2m_1 a + b) e^{m_1 x}$$

$$= 0$$

$am_1^2 + bm_1 + c = 0$

$m_1 = -b/2a$ ✓

Example Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Auxiliary equation $0 = m^2 + 2m + 1 = (m+1)^2$

$\Rightarrow m = m_1 = -1$

Just one solution $y_1(x) = e^{m_1 x} = e^{-x}$

Because repeated root general soln. is

$$y(x) = Ae^{m_1 x} + Bxe^{m_1 x} = (A + Bx)e^{-x}.$$

Example mechanical osallator

$$\frac{d^2x}{dt} + 2r\frac{dx}{dt} + \omega_0^2 x = 0$$

Auxiliary equation $m^2 + 2rm + \omega_0^2 = 0$

$$\Rightarrow m = -r \pm \sqrt{r^2 - \omega_0^2}$$

Case of critical damping $\omega_0 = r$

$m = -r$ repeated

General solution

$$x(t) = (A + Bt)e^{-rt}$$

Typical trajectory