DYNAMICS AND CONTROL

CONTROL SEMINAR 4

GENERAL INTRODUCTION – SEMINAR 4

- Response of 1st order system to ramp input; velocity lag
- Response of 1st order system to sine input: phase lag
- Response of 2nd order system to step and ramp inputs
- Introduction to the concept of root locus
- Example sheet 4 questions 1 and 2

Hydraulic Position Control System under Standard Inputs

ii) Ramp Input

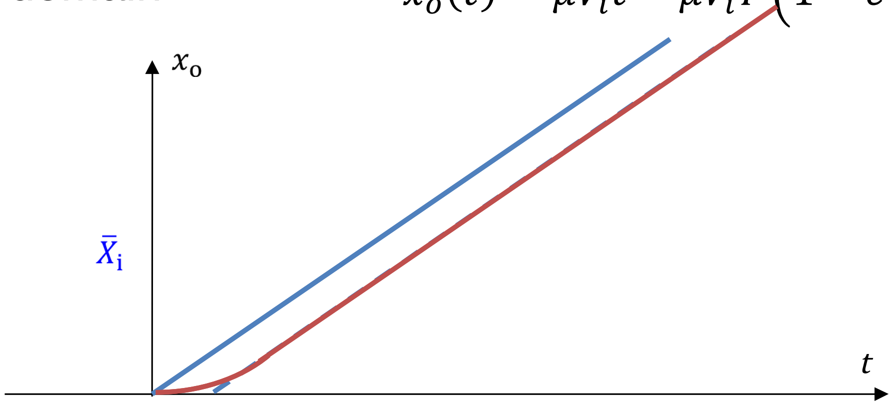
$t < 0 \qquad x_i(t) = 0$

$t \geq 0 \qquad x_i(t) = \bar{V}_i t$

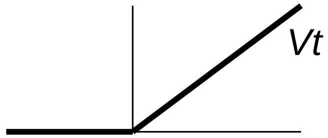
From the the table of L.T. $$X_i(s) = \frac{\bar{V}_i}{s^2}$$

The output in s-domain $$X_o(s) = \frac{\mu \bar{V}_i}{s^2(1 + Ts)}$$

In the time domain $$x_o(t) = \mu \bar{V}_i t - \mu \bar{V}_i T \left(1 - e^{-t/T}\right)$$



ii) ramp Input



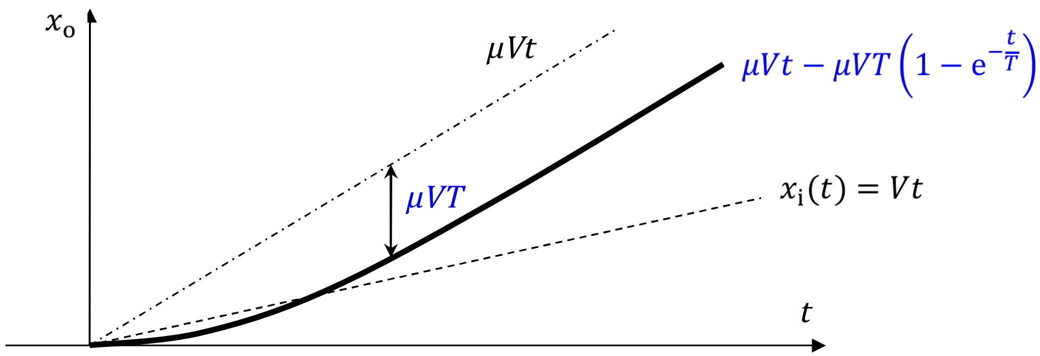
$t < 0 \qquad x_i(t) = 0$

$t \geq 0 \qquad x_i(t) = Vt$

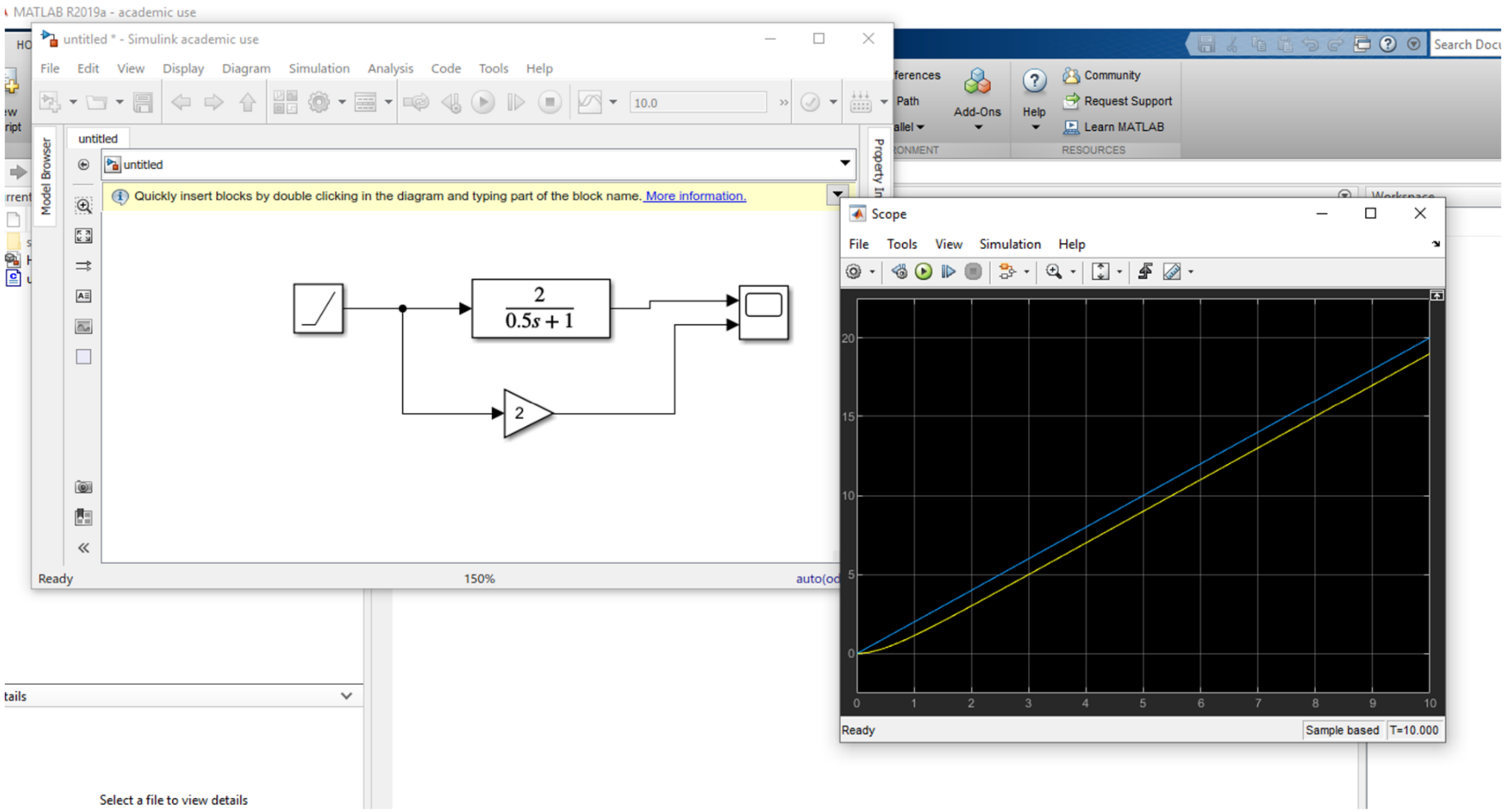
From the table of L.T. $$X_i(s) = \frac{V}{s^2} \qquad (14)$$

The output in s-domain $$X_o(s) = \frac{\mu V}{s^2(1 + Ts)} \qquad (15)$$

In the time domain $$x_o(t) = \mu Vt - \mu VT\left(1 - e^{-\frac{t}{T}}\right) \qquad (16)$$

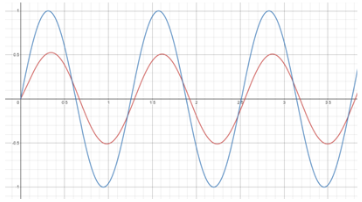


Simulink model



Hydraulic Position Control System under Standard Inputs

iii) Oscillatory input



$t < 0 \qquad x_i(t) = 0$

$t \geq 0 \qquad x_i(t) = A\sin(\omega t)$

From the table of L.T.

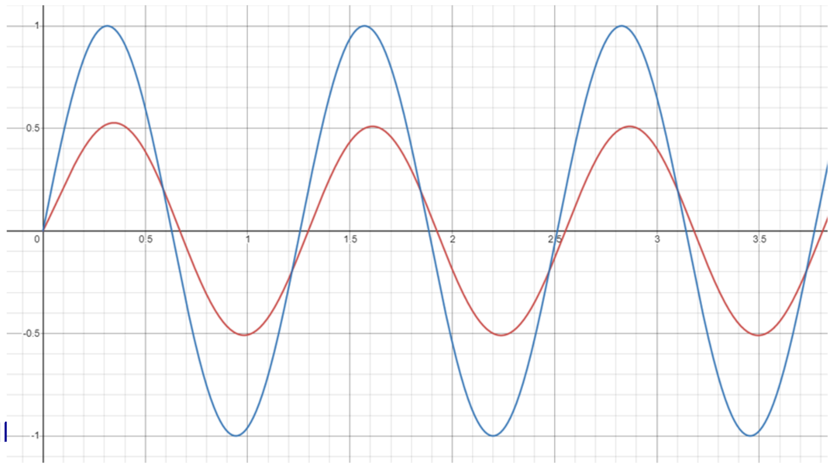
$$X_i(s) = \frac{A\omega^2}{(s^2 + \omega^2)} \qquad (5)$$

The output in s-domain

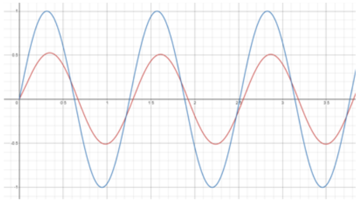
$$X_o(s) = \frac{\mu A\omega^2}{(s^2 + \omega^2)(1 + Ts)} \qquad (6)$$

$$X_{out}(s) = \frac{\mu A\omega^2}{1+\omega^2T^2}\left(\frac{1-Ts}{(s^2+\omega^2)} + \frac{T^2}{(1+Ts)}\right) = \frac{\mu A\omega^2}{1+\omega^2T^2}\left(\frac{1}{(s^2+\omega^2)} - \frac{Ts}{(s^2+\omega^2)} + \frac{T^2}{(1+Ts)}\right)$$

Steady state: gain and phase angle (1st order lag)



iii) Oscillatory input



$t < 0 \qquad x_i(t) = 0$

$t \geq 0 \qquad x_i(t) = A\sin(\omega t)$

From the table of L.T.

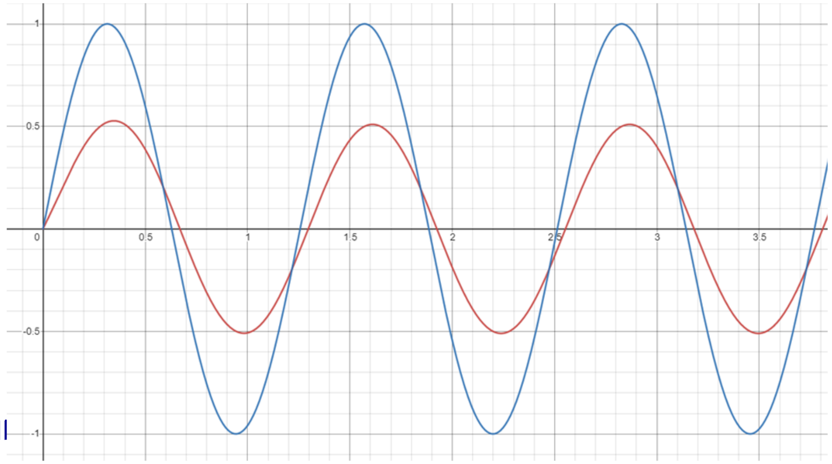
$$X_i(s) = \frac{A\omega^2}{(s^2 + \omega^2)} \tag{5}$$

The output in s-domain

$$X_o(s) = \frac{\mu A\omega^2}{(s^2 + \omega^2)(1 + Ts)} \tag{6}$$

$$X_{out}(s) = \frac{\mu A\omega^2}{1+\omega^2T^2}\left(\frac{1-Ts}{(s^2+\omega^2)} + \frac{T^2}{(1+Ts)}\right) = \frac{\mu A\omega^2}{1+\omega^2T^2}\left(\frac{1}{(s^2+\omega^2)} - \frac{Ts}{(s^2+\omega^2)} + \frac{T^2}{(1+Ts)}\right)$$

Steady state: gain and phase angle (1st order lag)

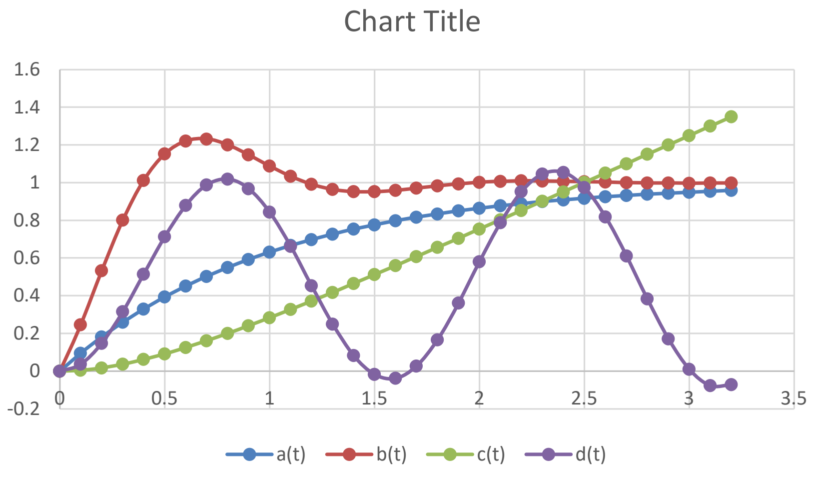


Recap: The Final Value Theorem

The **final value theorem:**

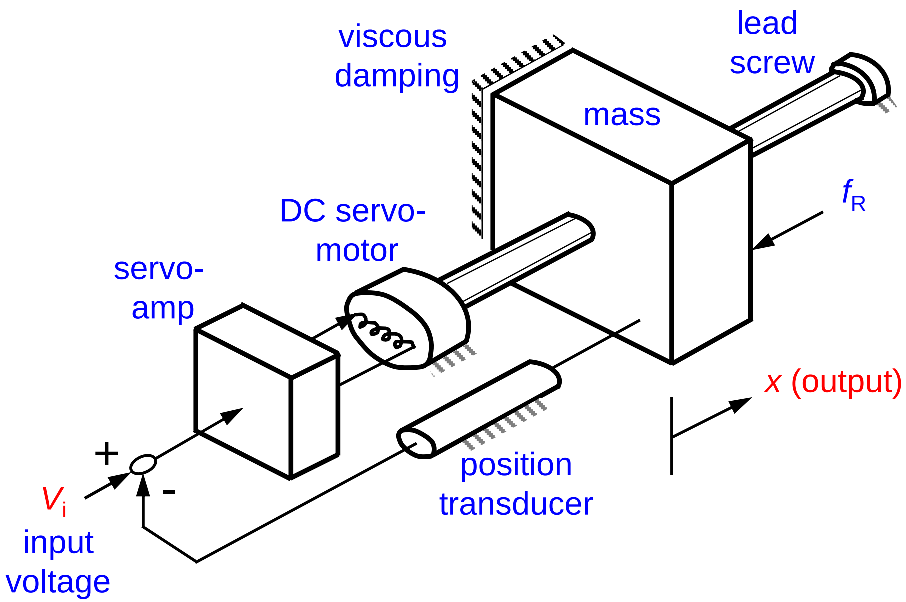
$$x_{ss} = \lim_{t\to\infty} x_o(t) = \lim_{s\to 0} sX_o(s) \qquad (9)$$

Gives the steady-state response of a system.
Some provisos:
Steady state implies that we have a finite end value:



Which of these can we use the finite value theorem on?
a(t)?
b(t)?
c(t)?
d(t)?

Example: Electro-Mechanical Position Control System



It will be shown that the **transfer functions** may be written as

$$\frac{X(s)}{X_i(s)} = \frac{\omega_n^2}{s^2 + 2\gamma\omega_n s + \omega_n^2}$$

$$\frac{X(s)}{F_R(s)} = \frac{-1}{M(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

2nd order system

https://www.youtube.com/watch?v=Sn8DqDGwazs

E.-M. Position Control System: Equations for the Model

i) **Position Transducer** output $V_x = K_4 x$ K_4 is constant

error voltage $V_e = V_i - V_x = V_i - K_4 x$

ii) **Servo-Amplifier** develops current (K_1 is another constant)

$$i_f = K_1 V_e = K_1 (V_i - K_4 x)$$

iii) **DC Servo-Motor** develops torque (K_2 is motor constant)

$$l_m = K_2 i_f = K_2 K_1 (V_i - K_4 x)$$

iv) At **Lead Screw** the torque is converted into a force on the load mass

$$f_m = K_3 l_m = K_3 K_2 K_1 (V_i - K_4 x) \qquad K_3 = 2\pi/(\text{pitch of leadscrew})$$

Laplace domain $F_m(s) = K_1 K_2 K_3 (V_i(s) - K_4 X(s))$ (1)

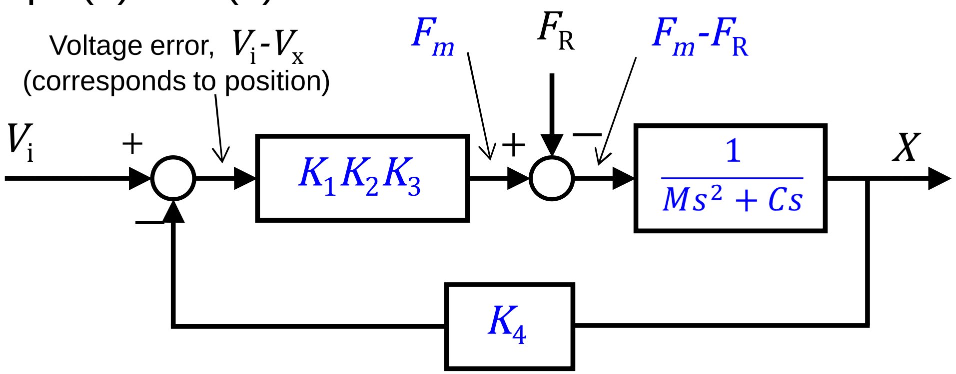
v) For the **Load Mass** assuming viscous damping

$$M\ddot{x} + C\dot{x} = f_m - f_\mathrm{R}$$

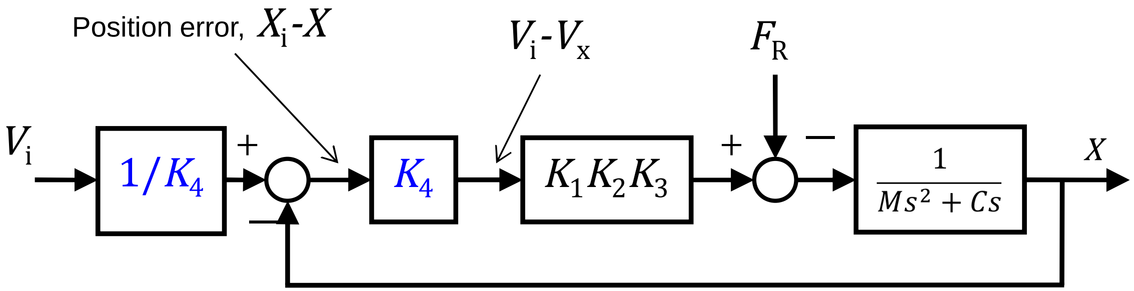
Laplace domain $X(s) = \dfrac{F_m(s) - F_\mathrm{R}(s)}{Ms^2 + Cs}$ (2)

E.-M. Position Control System: Block Diagrams

Using Eqs. (1) and (2)



Focusing on actual position error



with $K = K_1K_2K_3K_4$

$$X(s) = ([X_i(s) - X(s)]K - F_R(s))\frac{1}{Ms^2 + Cs}$$

E-.M. Position Control System: Overall Transfer Function

Rearranging $$[Ms^2 + Cs + K]X(s) = KX_{\mathrm{i}}(s) - F_{\mathrm{R}}(s) \qquad (3)$$

Preferred form

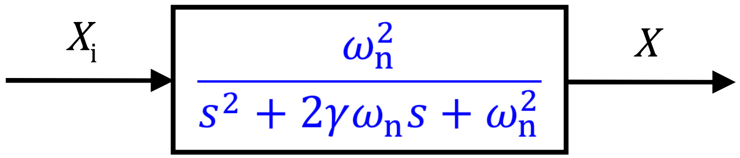
$$[s^2 + 2\gamma\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2]X(s) = \omega_{\mathrm{n}}^2 X_{\mathrm{i}}(s) - \frac{F_{\mathrm{R}}(s)}{M}$$

with $\dfrac{C}{M} = 2\gamma\omega_{\mathrm{n}}$ and $\omega_{\mathrm{n}}^2 = \dfrac{K}{M}$

$$X(s) = \frac{\omega_n^2 X_i(s)}{s^2 + 2\gamma\omega_n s + \omega_n^2} - \frac{F_R(s)}{M(s^2 + 2\gamma\omega_n s + \omega_n^2)}$$

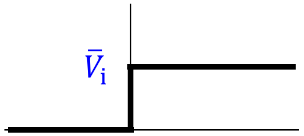
Transfer function

$$\frac{X(s)}{X_{\mathrm{i}}(s)} = \frac{\omega_{\mathrm{n}}^2}{s^2 + 2\gamma\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2} \qquad (4)$$



E.-M. Position Control System under Standard Inputs

i) step Input



$t < 0 \qquad V_{\mathrm{i}}(t) = 0$

$t \geq 0 \qquad V_{\mathrm{i}}(t) = \bar{V}_{\mathrm{i}}$

From the table of L.T. $$X_{\mathrm{i}}(s) = \frac{\bar{V}_{\mathrm{i}}}{K_4 s} = \frac{\bar{X}_{\mathrm{i}}}{s} \qquad (5)$$

The output in s-domain

$$X_{\mathrm{o}}(s) = \frac{\omega_{\mathrm{n}}^2 \bar{X}_{\mathrm{i}}}{s(s^2 + 2\gamma\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2)} = \frac{\omega_{\mathrm{n}}^2 \bar{X}_{\mathrm{i}}}{s(s - p_1)(s - p_2)} \qquad (6)$$

with the roots of the characteristic equation

$$s^2 + 2\gamma\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2 = 0$$

$$p_1 = -\gamma\omega_{\mathrm{n}} + \omega_{\mathrm{n}}\sqrt{\gamma^2 - 1} \qquad p_2 = -\gamma\omega_{\mathrm{n}} - \omega_{\mathrm{n}}\sqrt{\gamma^2 - 1}$$

E.-M. Position Control System under Step Input

Assuming a *unit step input* and using partial fractions

$$X_o(s) = \frac{B}{s} + \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2}$$

where (for $\gamma \neq 1$)

$$B = 1\,; \;\; A_1 = -\frac{1}{2} - \frac{\gamma}{2\sqrt{\gamma^2 - 1}}\,; \;\; A_2 = -\frac{1}{2} + \frac{\gamma}{2\sqrt{\gamma^2 - 1}}$$

With the inverse Laplace transform, in the time domain

$$x_o(t) = B + A_1 e^{p_1 t} + A_2 e^{p_2 t} \tag{7}$$

This solution, valid for $\gamma \neq 1$, gives rise to two distinct types of transient response.

E.-M. Position Control System under Step Input

i) $\gamma > 1$ — p_1 and p_2 are **real** and **unequal**. For this situation the response is overdamped (non-oscillatory).

ii) $\gamma < 1$ — p_1 and p_2 are **complex conjugate** (as A_1 and A_2)

$$p_1 = -\gamma\omega_\mathrm{n} + \mathrm{i}\omega_\mathrm{n}\sqrt{1-\gamma^2}$$

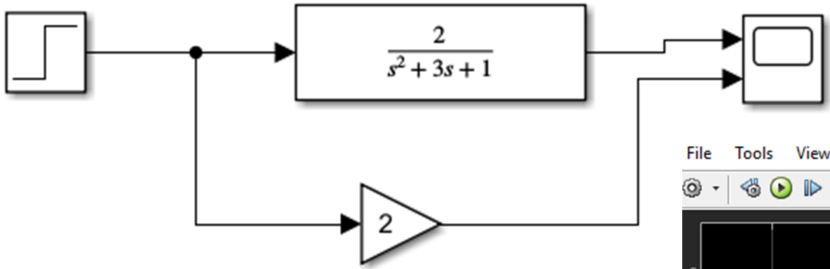
$$p_2 = -\gamma\omega_\mathrm{n} - \mathrm{i}\omega_\mathrm{n}\sqrt{1-\gamma^2}$$

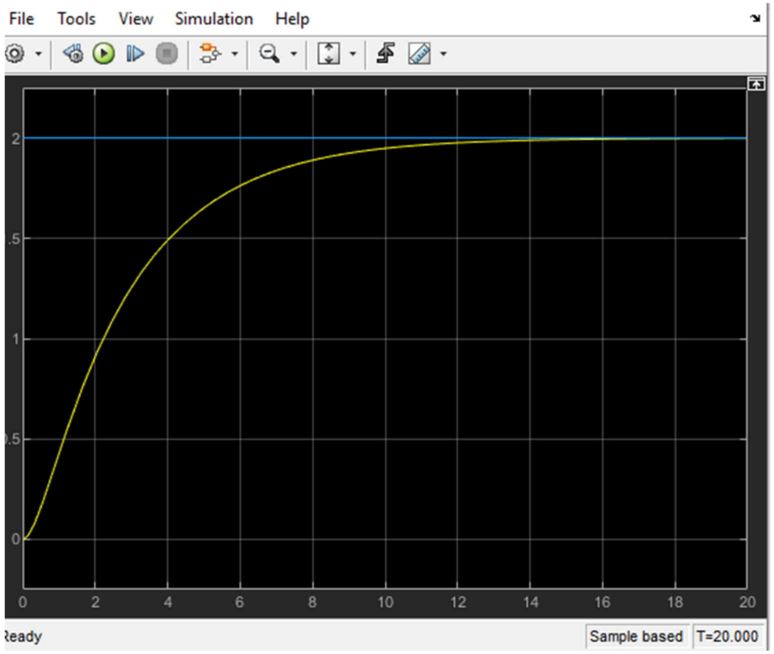
$$x_\mathrm{o}(t) = \bar{X}_\mathrm{i}\left[1 - \frac{\mathrm{e}^{-\gamma\omega_\mathrm{n}t}}{\sqrt{1-\gamma^2}}\sin(\omega_\mathrm{n}t\sqrt{1-\gamma^2} + \phi)\right]$$

Maximum overshoot at $$t = \frac{\pi}{\omega_\mathrm{n}\sqrt{1-\gamma^2}}$$

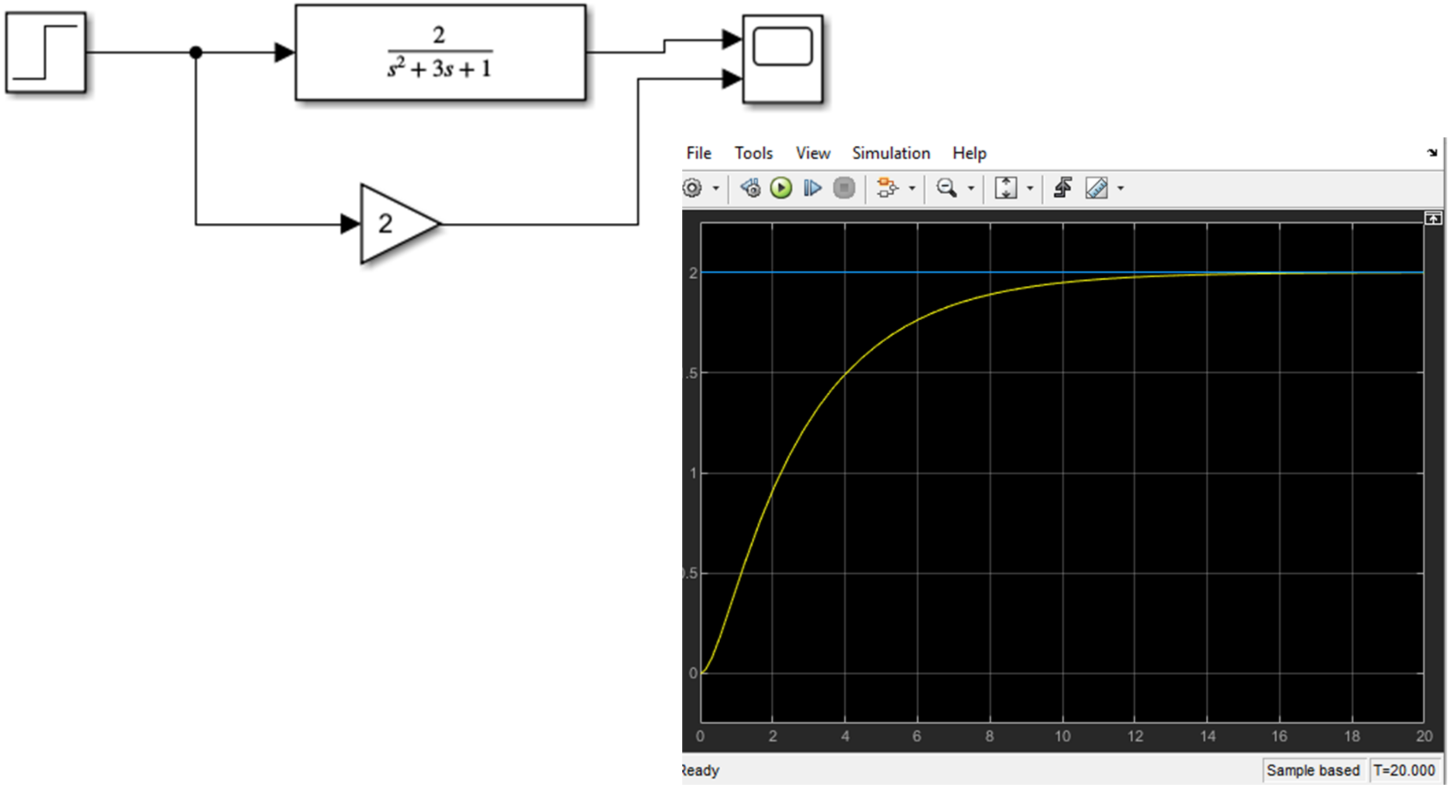
with magnitude $$x_\mathrm{max} = \bar{X}_\mathrm{i}\left(1 + \mathrm{e}^{\frac{-\gamma\pi}{\sqrt{1-\gamma^2}}}\right)$$

Simulink Model: γ>1



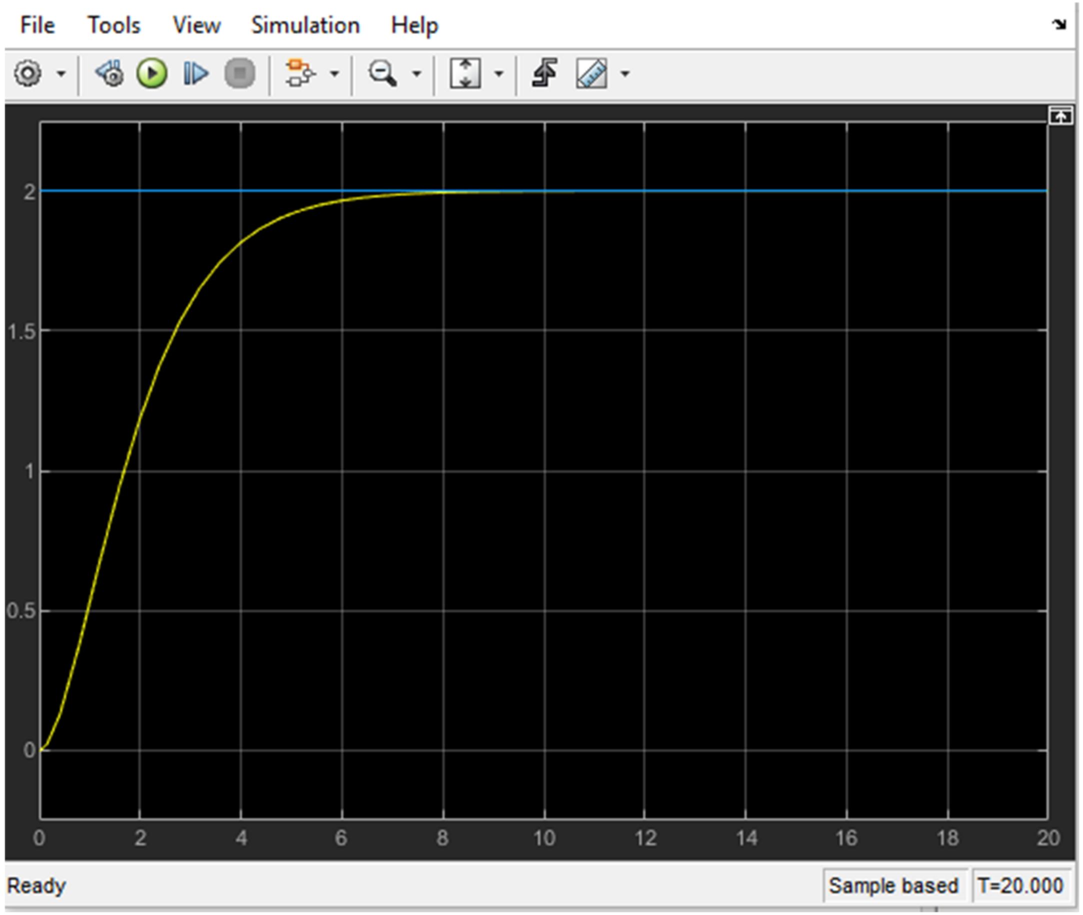


Simulink Model: γ>1



Simulink Model: $\gamma=1$

What is the transfer function in this case?

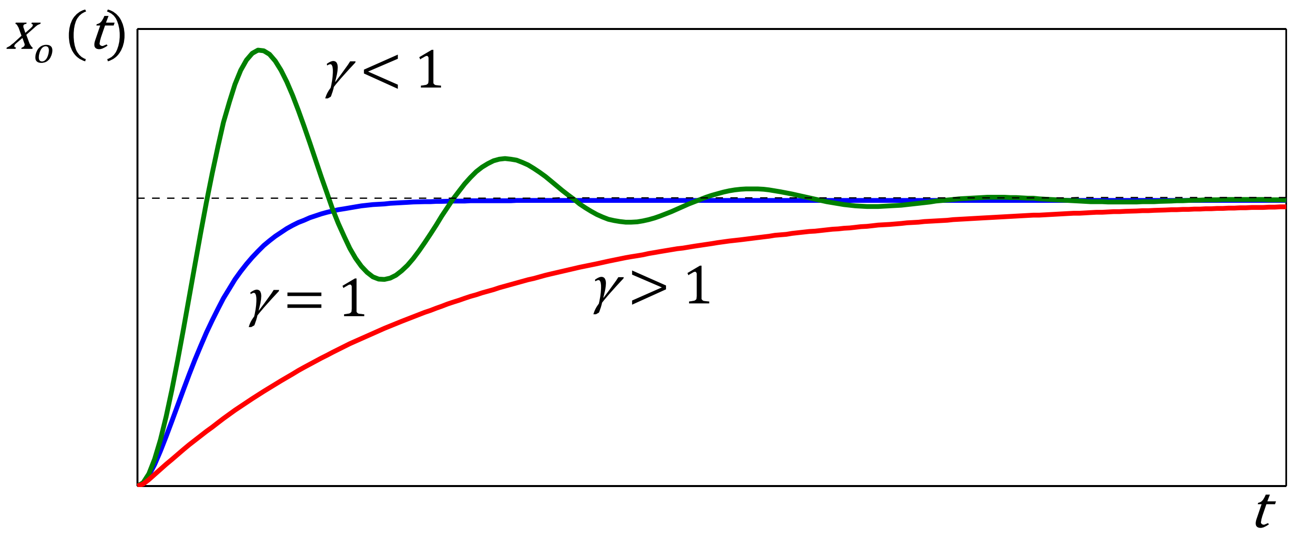


E.-M. Position Control System under Step Input

iii) $\gamma = 1$ p_1 and p_2 are **real** and **equal** (= - ω_n) and the response is Is said to be critically damped.

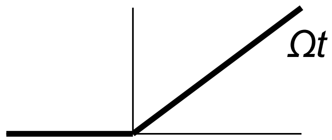
$$x_o(t) = \bar{X}_i[1 - (1 + \omega_n t)e^{-\omega_n t}]$$

The transient responses under a step input for all three cases can be summarised



E.-M. Position Control System under Standard Inputs

ii) ramp Input



$t < 0 \qquad V_{\mathrm{i}}(t) = 0$

$t \geq 0 \qquad V_{\mathrm{i}}(t) = \Omega t$

From the table of L.T. $$V_{\mathrm{i}}(s) = \frac{\Omega}{s^2}$$

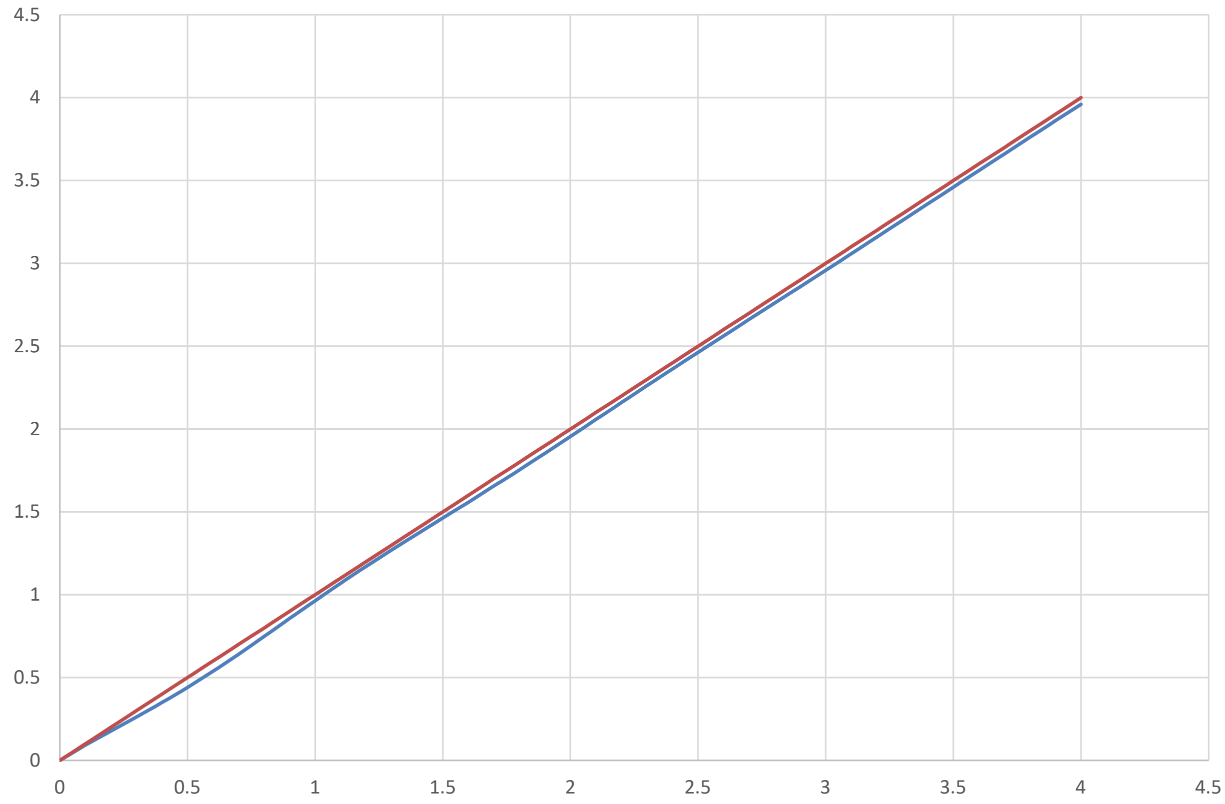
and from the b.d. $$X_{\mathrm{i}}(s) = \frac{V_{\mathrm{i}}(s)}{K_4} = \frac{\Omega}{s^2 K_4} = \frac{\Omega_x}{s^2} \qquad (8)$$

The output in s-domain $$X_{\mathrm{o}}(s) = \frac{\omega_{\mathrm{n}}^2 \Omega_x}{s^2(s^2 + 2\gamma\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2)} \qquad (9)$$

In the time domain

$$x_{\mathrm{o}}(t) = \Omega_x \left(t - \frac{2\gamma}{\omega_{\mathrm{n}}} + A_1 \mathrm{e}^{p_1 t} + A_2 \mathrm{e}^{p_2 t} \right) \qquad (10)$$

Output $x_o(t)$



E.-M. Position Control System: S.-S. Error under Ramp Input

From the block diagram, for $F_R = 0$ (no disturbance)

$$E(s) = X_i(s) - X_o(s) = \frac{Ms^2 + Cs}{Ms^2 + Cs + K} X_i(s) \qquad (11)$$

For a ramp input $X_i(s)$ from Eq. (8)

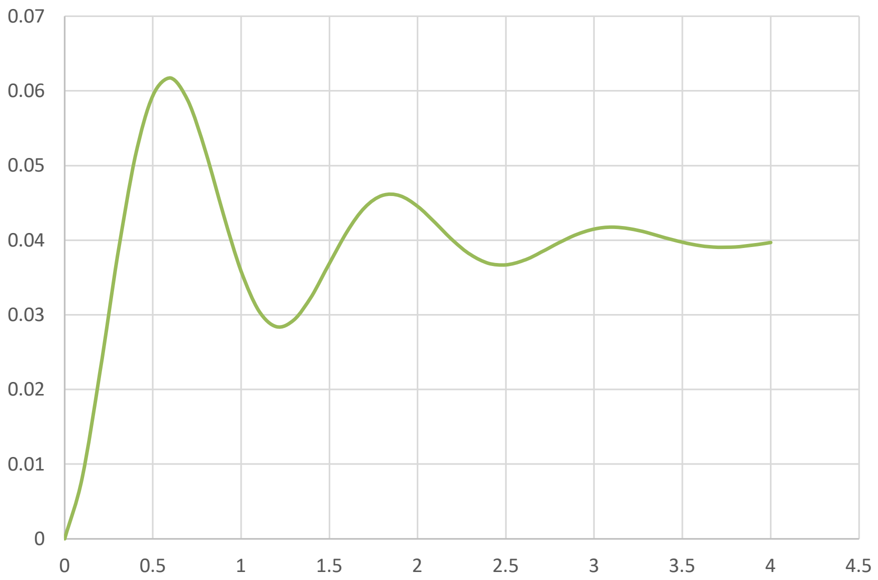
$$E(s) = \frac{Ms^2 + Cs}{Ms^2 + Cs + K} \frac{\Omega_x}{s^2} = \frac{Ms + C}{Ms^2 + Cs + K} \frac{\Omega_x}{s} \qquad (12)$$

Using the **final value theorem** the steady-state error

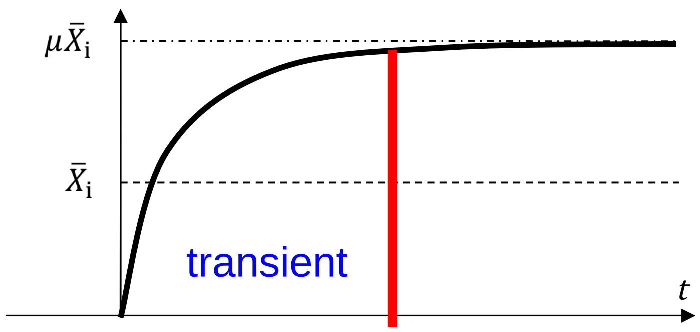
$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{Ms + C}{Ms^2 + Cs + K} \Omega_x$$

$$= \frac{C}{K} \Omega_x = \frac{2\gamma}{\omega_n} \Omega_x \qquad (13)$$

e(t) for γ<1



Hydraulic Position Control System: Transient Response



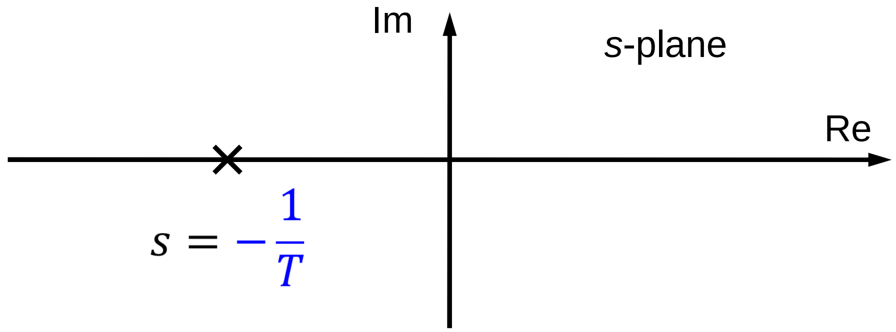
characteristic equation

Set denominator =0

$$P(s) = s + \frac{1}{T} = 0$$

with one real root $s = -\frac{1}{T}$

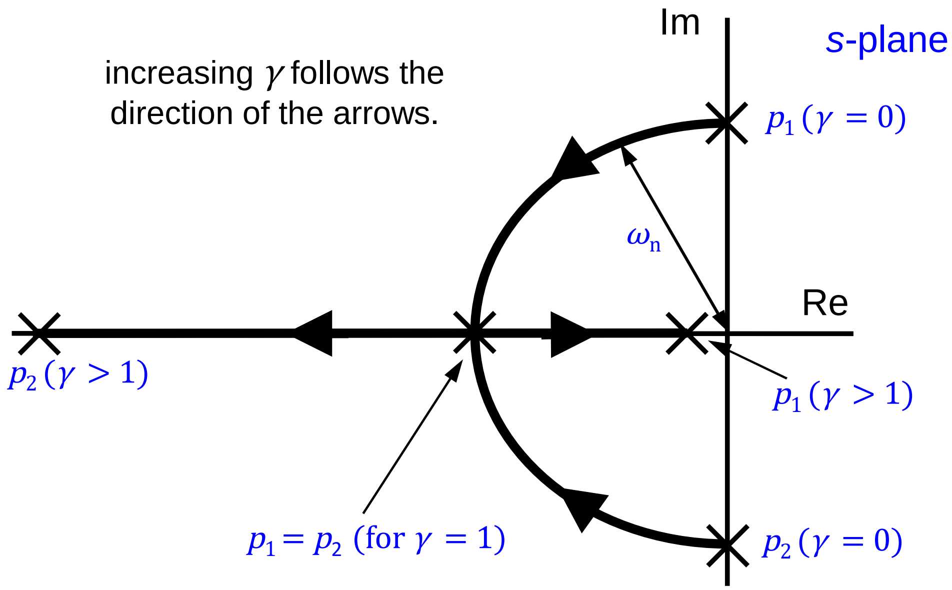
The **roots of the C.E.** in the **s-plane** govern stability and transient behaviour



E.-M. Position Control System: Transient Response

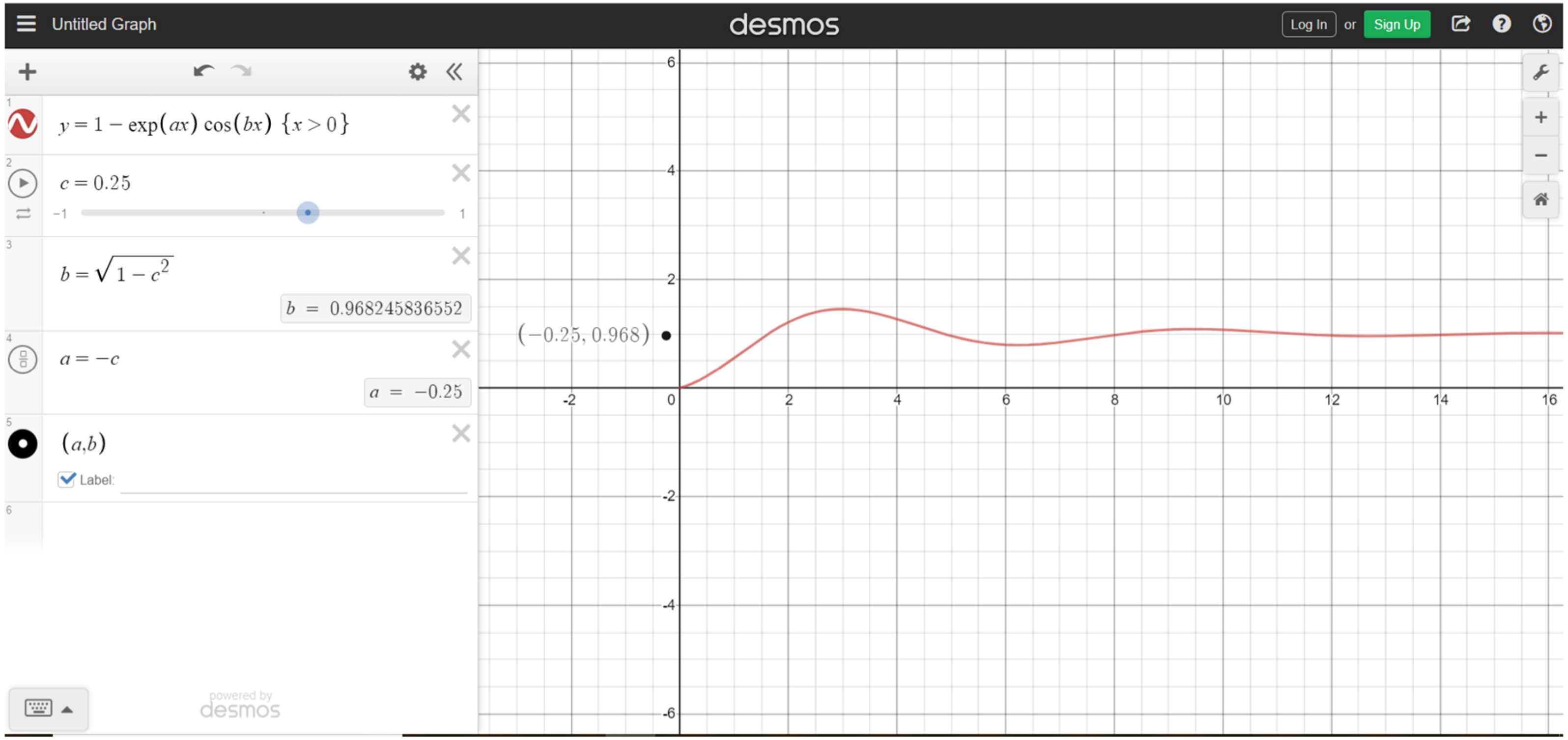
The **roots of the C.E.** in the **s-plane** govern stability and transient behaviour

$$p_1 = -\gamma\omega_n + \omega_n\sqrt{\gamma^2 - 1} \qquad p_2 = -\gamma\omega_n - \omega_n\sqrt{\gamma^2 - 1}$$



The roots trace out **loci** in the s-plane.

Visualisation of root locus



Example sheet 4 question 1

1. Figure 1 shows a mass-damper-spring system with an applied force $p(t)$.
   a. Derive the transfer function $G(s)$ that relates the applied force $p(t)$ to the velocity of the mass, $v(t)$. Let the Laplace Transform of $p(t)$ and $v(t)$ to be $P(s)$ and $V(s)$, respectively.
   b. Determine the steady state velocity response of the mass when a step input force is applied to the system. The magnitude of the step input is α.
   c. Determine the steady state velocity response of the mass when a ramp input force $p(t)=\alpha t$, is applied to the system.

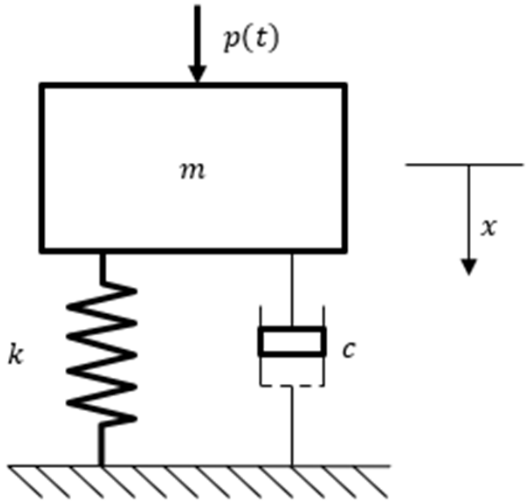


Figure 1.

Example sheet 4 question 1

a) Derive the transfer function *G(s)* that relates the applied force *p(t)* to the velocity of the mass, *v(t)*. Let the Laplace Transform of *p(t)* and *v(t)* to be *P(s)* and *V(s)*, respectively.

The first step here is to determine the equation of motion in the time domain: if the velocity of the mass is $\dot{x}$ then the force due to the damper is $-c\dot{x}$ (note that it will always oppose the motion). Force due to the spring is $-kx$, so the net force acting on the mass will be:

$$\text{Net force} = p(t) - c\dot{x} - kx$$

Therefore if the acceleration of the mass is $\ddot{x}$:

$$m\ddot{x} = p(t) - c\dot{x} - kx$$

Rearranging gives the familiar form:

$$p(t) = m\ddot{x} + c\dot{x} + kx$$

And Laplace transforms give us:

$$P(s) = (ms^2 + cs + k)X(s)$$

Example sheet 4 question 1

a) Derive the transfer function *G(s)* that relates the applied force *p(t)* to the velocity of the mass, *v(t)*. Let the Laplace Transform of *p(t)* and *v(t)* to be *P(s)* and *V(s)*, respectively.

Laplace transforms give us:

$$P(s) = (ms^2 + cs + k)X(s)$$

This is fine – but the question asks for a transfer function in terms of the velocity, v. I find it easiest to work in terms of x to here, and then to substitute as follows:

If $v = \dot{x}$, then for a system that is initially at rest (number 1 in the table of Laplace transforms):

$$V(s) = sX(s)$$

So substituting ${}^{V(s)}/_{s}$ for $X(s)$:

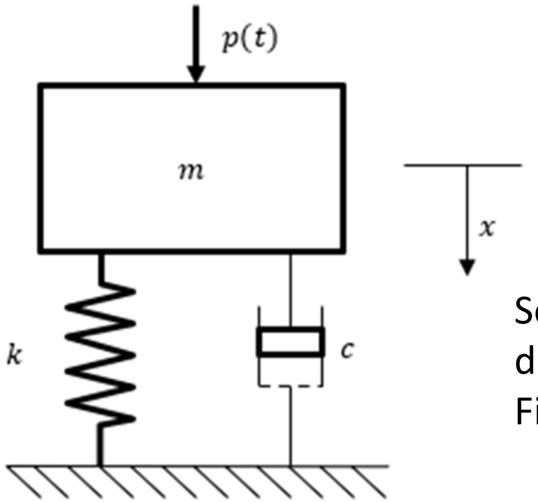
$$P(s) = (ms^2 + cs + k)X(s) = \frac{(ms^2 + cs + k)V(s)}{s}$$

Rearranging gives the transfer function:

$$G(s) = \frac{V(s)}{P(s)} = \frac{s}{ms^2 + cs + k}$$

Example sheet 4 question 1

b. Determine the steady state velocity response of the mass when a step input force is applied to the system. The magnitude of the step input is a.



$$G(s) = \frac{V(s)}{P(s)} = \frac{s}{ms^2 + cs + k}$$

$$P(s) = \frac{a}{s}$$

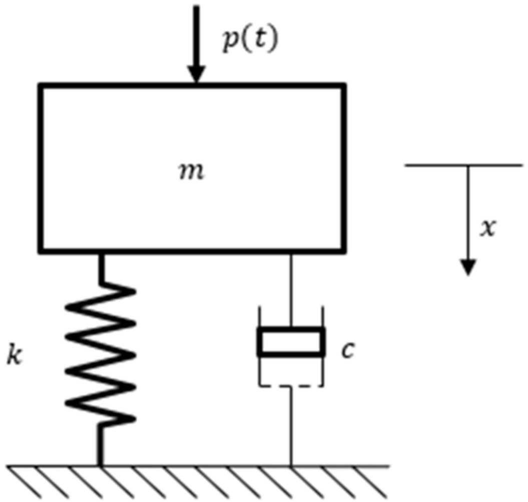
$$V(s) = \frac{as}{s\,(ms^2 + cs + k)} = \frac{a}{ms^2 + cs + k}$$

See worked example solutions for how to do this in the time domain … it's rather complicated.

Final value theorem:

$$\lim_{t \to \infty} v(t) = \lim_{s \to 0} sV(s) = \frac{as}{ms^2 + cs + k} = 0$$

Example sheet 4 question 1



(c) Determine the steady state velocity response of the mass when a ramp input force $p(t) = \sigma t$, is applied to system.

From the table of Laplace transforms (no. 6, multiply by σ):

$$P(s) = \frac{\sigma}{s^2}$$

$$V(s) = G(s)P(s) = \frac{s\sigma}{s^2(ms^2 + cs + k)}$$

$$V(s) = \frac{\sigma}{s(ms^2 + cs + k)}$$

Using the final value theorem:

$$\lim_{t\to\infty} v(t) = \lim_{\substack{t \\ s\to 0}} sV(s) = \frac{s\sigma}{s(ms^2 + cs + k)} = \frac{\sigma}{ms^2 + cs + k} = \frac{\sigma}{k}$$

Top tip: be comfortable using the final value theorem. It saves time and gets the same marks!

Example sheet 4 question 2

2. For the system described in Q1, a control system is designed to regulate the velocity of the mass, using a proportional controller, *Kc(s)=K,* with a reference velocity *vR(t)*. The block diagram representation of the control system is shown in Figure 2. There are two different forces applied to the mass: the disturbance force, *fd(t),* and the control force, *fc(t).*
   a. Determine the transfer function from the reference velocity *VR(s)* to the velocity of the mass *V(s)*. Draw the corresponding block diagram.
   b. Determine the transfer function from the disturbance force *Fd(s)* to the velocity of the mass *V(s)*. Draw the corresponding block diagram.
   c. What is the effect of the proportional control gain to the system damping?

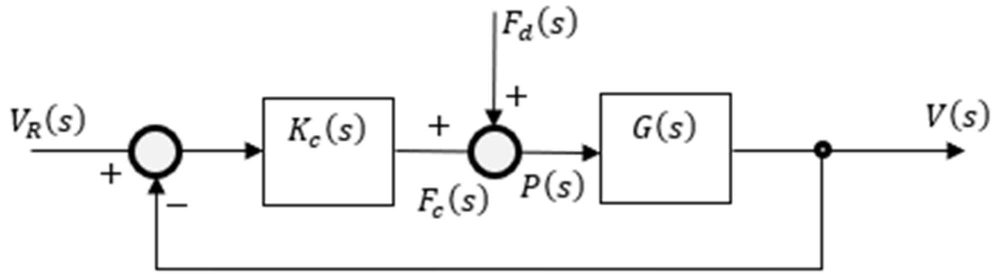
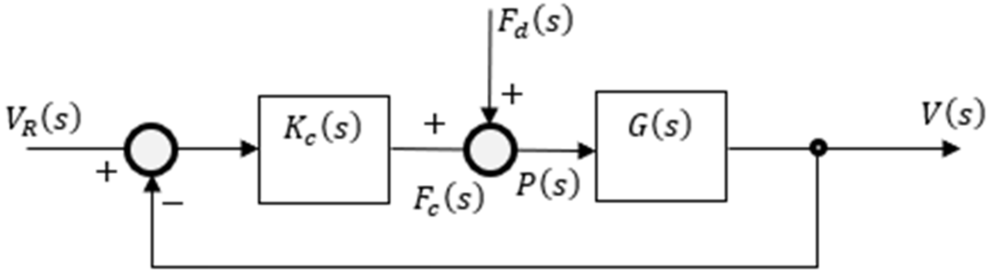


Figure 2.

Example sheet 4 question 2



(a)

$$G(s) = \frac{V(s)}{P(s)} = \frac{s}{ms^2 + cs + k}$$

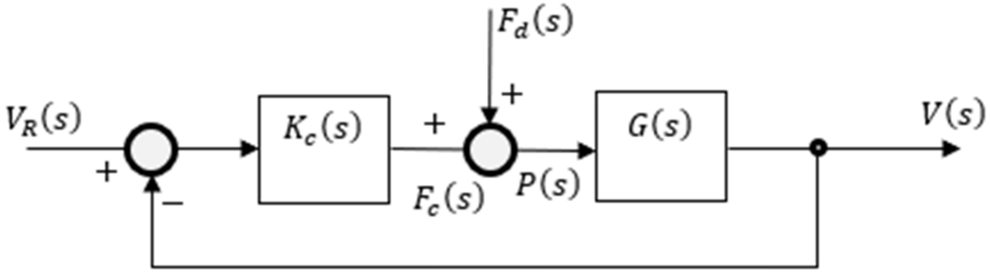
$$V(s) = \big(V_R(s) - V(s)\big)K_c(s)G(s)$$

$$V(s)\big(1 + K_c(s)G(s)\big) = V_R(s)K_c(s)G(s)$$

$$\frac{V(s)}{V_R(s)} = \frac{K_c(s)G(s)}{1 + K_c(s)G(s)} = \frac{sK_c(s)}{ms^2 + cs + k + sK_c(s)}$$

$$\frac{V(s)}{V_R(s)} = \frac{K_c(s)G(s)}{1 + K_c(s)G(s)} = \frac{sK_c(s)}{ms^2 + \big(c + K_c(s)\big)s + k}$$

Example sheet 4 question 2



(b)

$$G(s) = \frac{s}{ms^2 + cs + k}$$

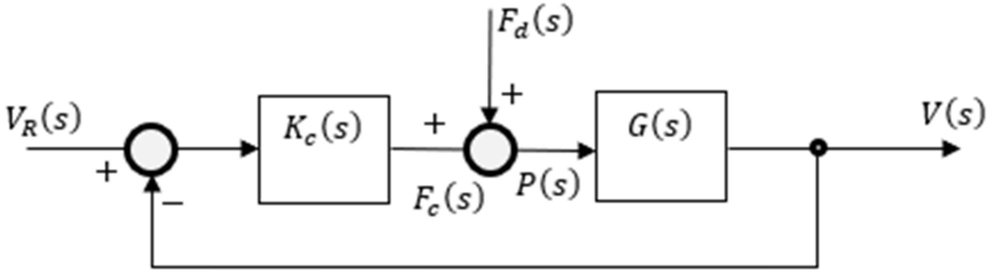
$$V(s) = \big(F_d(s) - K_c(s)V(s)\big)G(s)$$

$$V(s)\big(1 + K_c(s)G(s)\big) = F_d(s)G(s)$$

$$\frac{V(s)}{F_d(s)} = \frac{G(s)}{1 + K_c(s)G(s)} = \frac{s}{ms^2 + cs + k + sK_c(s)}$$

$$\frac{V(s)}{F_d(s)} = \frac{s}{ms^2 + \big(c + K_c(s)\big)s + k}$$

Example sheet 4 question 2



(c) What is the effect of the proportional gain K_c on the system damping?

$$\frac{s}{ms^2 + cs + k}$$

Characteristic equation:

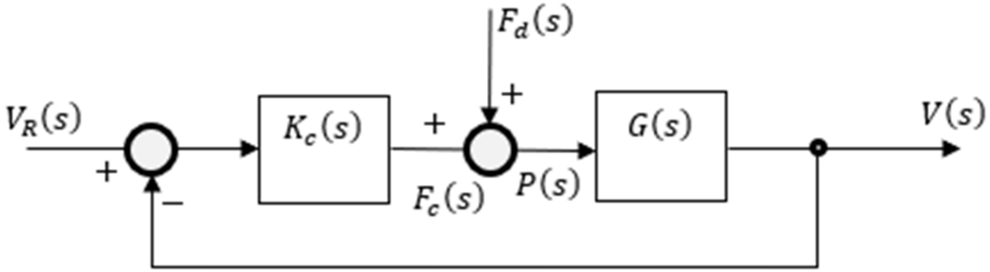
$$ms^2 + \left(c + K_c(s)\right)s + k = m(s^2 + 2\gamma\omega_n s + \omega_n^2) = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$2\gamma\omega_n = \frac{\left(c + K_c(s)\right)}{m}$$

Example sheet 4 question 2



(c)

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$2\gamma\omega_n = \frac{\left(c + K_c(s)\right)}{m}$$

$$\gamma = \frac{\left(c + K_c(s)\right)}{2\sqrt{km}}$$

And therefore increasing the proportional gain $K_c(s)$ will have the effect of making the system more stable.

The End …
Next week in Dynamics and Control

LECTURE 5 – PID CONTROLLERS, STABILITY IN HIGHER ORDER SYSTEMS