University of Nottingham
Department of Mechanical Engineering

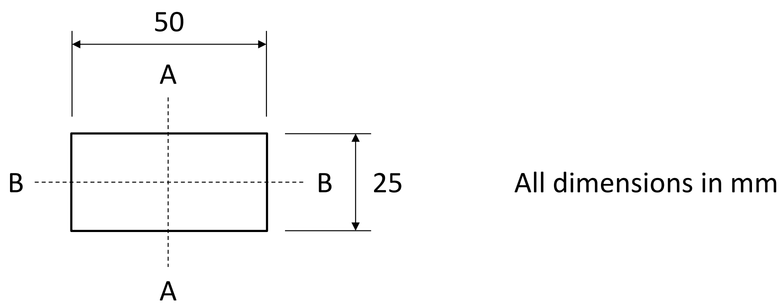
MMME2053 Mechanics of Solids
Exercise Sheet 9 – Elastic Instability (Buckling)

1. Find the critical load for a steel column of rectangular cross-section, 50mm by 25mm and of length 3m for each of the following assumptions:

    (a) Both ends are hinged
    (b) Both ends are built-in (fixed)
    (c) One end is hinged and the other end is built-in (fixed).

[Ans: a) 14.28kN, b) 57.12kN, c) 29.2kN]

Solution 1



$$I_{min} = I_{BB} = \frac{bd^3}{12} = \frac{50 \times 25^3}{12} = 65{,}104.17mm^4$$

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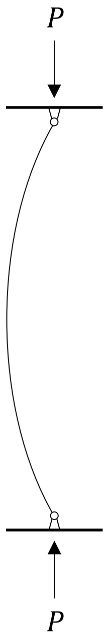
(a) **Hinged-hinged**



$$P_{crit} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 200{,}000 \times 65{,}104.17}{3000^2} = 14{,}278.94N$$

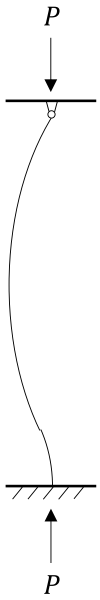
$$\boxed{\therefore P_{crit} = 14.28kN}$$

(b) **Fixed-fixed**

$$P_{crit} = \frac{4\pi^2 EI}{l^2} = \frac{4 \times \pi^2 \times 200{,}000 \times 65{,}104.17}{3000^2} = 57{,}115.76N$$

$$\boxed{\therefore P_{crit} = 57.12kN}$$

(c) **Fixed-hinged**



$$P_{crit} = \frac{2.045\pi^2 EI}{l^2} = \frac{2.045 \times \pi^2 \times 200{,}000 \times 65{,}104.17}{3000^2} = 29{,}200.43N$$

$$\boxed{\therefore P_{crit} = 29.2kN}$$

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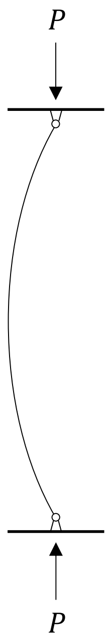
2. By what factor would the cross-section dimensions of the beam in question 1 have to be increased to ensure the same critical load for condition (a) if the material were changed to Aluminium with a Young's modulus of 70GPa?

[Ans: 30%]

Solution 2

For a hinged-hinged beam:



$$P_{crit} = \frac{\pi^2 EI}{l^2}$$

It is known that:

$$P_{crit} = 14{,}278.94N$$

$$\therefore \frac{\pi^2 EI}{l^2} = 14{,}278.94N \tag{1}$$

For Aluminium:

$$E = 70GPa$$

Therefore, from (1):

$$\therefore I = \frac{14{,}278.94 \times 3000^2}{\pi^2 \times 70{,}000} = 186{,}011.88mm^4$$

Ratio of new 2nd moment of area to initial 2nd moment of area is:

$$\frac{186{,}011.88}{65{,}104.17} = 2.857$$

As 2nd moment of area is a 4th order value (mm^4), the factor of increase required in the cross-sectional dimensions:

$$\sqrt[4]{2.857} = 1.3$$

The required increase in cross-sectional dimensions therefore is 30%.

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3. A new tripod for a surveying instrument will consist of three equal tubular legs, hinged at the top and resting on points at the bottom. When the instrument is set up for use, the top hinge points are equally spaced on a 100mm diameter horizontal circle, whilst the pointed ends are equally spaced on a 1m diameter horizontal circle. The top hinge points are 1.3m above the level ground on which the point ends rest. The greatest expected instrument weight is 80N. Assuming a factor of safety of 10 against elastic buckling, select the lightest safe Aluminium tube (E = 70GPa) from the following stock list (Table 3.1).

Table 3.1

Thickness (mm)	Outer diameters available (mm)
1	5, 10, 15, 20, 25, 30
2	10, 15, 20, 25, 30, 40
3	15, 20, 25, 30, 40, 50

[Ans: t=1mm & OD=15mm]

Solution 3

Considering one of the tripod legs:

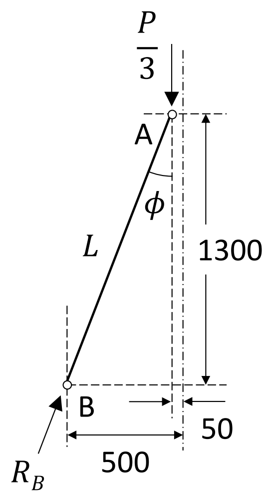

Fig 3.1

$$\therefore \phi = \tan^{-1}\left(\frac{500-50}{1300}\right) = 19.09°$$

and,

$$L = \sqrt{1300^2 + (500 - 50)^2} = 1375.68mm$$

Calculation of R_B:

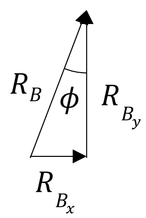

Fig 3.2

From Figs 3.1 and 3.2, vertical equilibrium gives:

$$\frac{P}{3} = R_{B_y} \tag{1}$$

and,

$$cos\phi = \frac{R_{B_y}}{R_B} \tag{2}$$

Rearranging (2) for R_{B_y} and substituting this into (1) gives:

$$\frac{P}{3} = R_B cos\phi$$

$$\therefore R_B = \frac{P}{3cos\phi} = \frac{80}{3cos(19.09)} = 28.22N$$

28.22N is therefore the maximum axial load. Applying a safety factor of 10 gives a maximum allowable design load (P_c) of 282.2N.

For a hinged-hinged beam:

$$P_c = \frac{\pi^2 EI}{L^2}$$

$$\therefore I = \frac{P_c \times L^2}{\pi^2 \times E} = \frac{282.2 \times 1375.68^2}{\pi^2 \times 70{,}000} = 773.03mm^4$$

Where for a hollow circular cross section beam:

$$I = \frac{\pi}{64}(D^4 - d^4) \tag{3}$$

Therefore, the calculation of I from (3) must give at least 773.03mm^4.

Substituting all values of D and t given in table 3.2 gives the results shown in table 3.2.

Table 3.2. Table of d, I and V values for ranges of D and t values (D = outer diameter (mm), t = thickness (mm), d = inner diameter (mm), I = second moment of area (mm^4), V = volume (mm^3)).

		D											
		5		10		15		20		25		30	
t	1	***d***	3	***d***	8	***d***	13	***d***	18	***d***	23	***d***	28
		I	26.70	***I***	289.81	***I***	1083.06	***I***	2700.98	***I***	5438.10	***I***	9588.93
		V	12.57	***V***	28.27	***V***	**43.98**	***V***	59.69	***V***	75.40	***V***	91.11
	2	***d***	1	***d***	6	***d***	11	***d***	16	***d***	21	***d***	26
		I	30.63	***I***	427.26	***I***	1766.36	***I***	4636.99	***I***	9628.20	***I***	17329.03
		V	18.85	***V***	50.27	***V***	81.68	***V***	113.10	***V***	144.51	***V***	175.93
	3			***d***	4	***d***	9	***d***	14	***d***	19	***d***	24
				I	478.31	***I***	2162.99	***I***	5968.24	***I***	12777.64	***I***	23474.77
				V	65.97	***V***	113.10	***V***	160.22	***V***	207.35	***V***	254.47

The options which give a high enough value of I are shown highlighted. Whichever one of these has the lowest corresponding V value is therefore the lightest possible option. This option is:

$$\boxed{D = 15\text{mm and } t = 1\text{mm}}$$

Note that the option for D = 5mm and t = 3mm is blocked out, this is as it is an unviable option as the thickness is too high to allow this value of D.

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Example calculations for Table 3.2

These example calculations correspond to the solution to the question option (D = 15mm and t = 1mm):

$$d = D - 2t = 15 - (2 \times 1) = 13mm$$

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(15^4 - 13^4) = 1083.06mm^4$$

$$V = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(15^2 - 13^2) = 43.98mm^2$$

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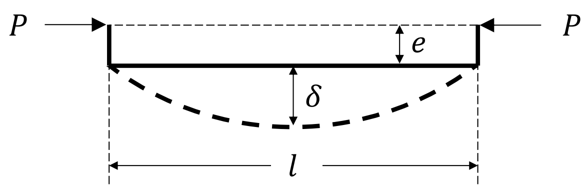
4. A hollow steel column, 9m long and of circular cross-section, 90mm outer diameter and 75mm inner diameter is subjected to an end thrust of 30kN. The line of action of thrust is parallel to the unstrained line of a strut, but does not coincide with it. Under load the maximum deviation of the strut from the straight line is 80mm. Find (a) the eccentricity of the load and (b) the maximum compressive and tensile stresses. The ends may be assumed to be hinged.

[Ans: a) 22.49mm, b) -98.41MPa & 67.55MPa]

Solution 4



(a)

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(90^4 - 75^4) = 1{,}667{,}467.8mm^4$$

For an eccentrically loaded beam:

$$\delta = e\left(sec\left(\frac{\alpha l}{2}\right) - 1\right) \quad (1)$$

and,

$$\alpha^2 = \frac{P}{EI} \quad (2)$$

As the column is made of Steel, assume:

$$E = 200GPa$$

Therefore, from (2):

$$\alpha^2 = \frac{P}{EI} = \frac{30{,}000}{200{,}000 \times 1{,}667{,}467.8} = 8.996 \times 10^{-8}$$

$$\therefore \alpha = 2.999 \times 10^{-4}$$

Trig rule:

$$sec = \frac{1}{cos}$$

Substituting this into (1) gives:

$$\delta = e\left(\frac{1}{cos\left(\frac{\alpha l}{2}\right)} - 1\right)$$

$$\therefore e = \frac{\delta}{\left(\frac{1}{cos\left(\frac{\alpha l}{2}\right)} - 1\right)} = \frac{80}{\left(\frac{1}{cos\left(\frac{2.999 \times 10^{-4} \times 9000}{2}\right)} - 1\right)}$$

$$\boxed{\therefore e = 22.49mm}$$

(b)

Stress will have an axial component and a bending component (due to the eccentricity of the load), therefore:

$$\sigma = \frac{P}{A} + \frac{My}{I} \quad (3)$$

where,

$$A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(90^2 - 75^2) = 1943.86mm^2$$

and,

$$M_{max} = P \times \pm(e + \delta) = 30{,}000 \times \pm(22.49 + 80) = \pm 3{,}074{,}700Nmm$$

Therefore, from (3):

$$\sigma_{max} = \frac{-30{,}000}{1943.86} \pm \frac{3{,}074{,}700 \times 45}{1{,}667{,}467.8}$$

Therefore,

$$\sigma_{max}^{compressive} = -15.43 - 82.98a$$

$$\boxed{\therefore\ \sigma_{max}^{compressive} = -98.41MPa}$$

and,

$$\sigma_{max}^{tensile} = -15.43 + 82.98$$

$$\boxed{\sigma_{max}^{tensile} = 67.55MPa.}$$

5. A steel strut with hinged ends, length 1.5m and cross-section 25mm by 25mm square, is found to be curved before applying a load. The curvature can be represented by

$$y_o = 5sin\left(\frac{2\pi x}{3}\right)mm$$

where y_o is the distance of a point on the axis of the strut in the unloaded state from the line of action of the loads which act through the hinge points, and xm is the distance of this point from one end of the strut. Find the maximum compressive stress if the loading is 15kN.

[Ans: -84.66MPa]

Solution 5

For a hinged-hinged beam with initial curvature (see page 14 of lecture notes):

$$y_o = \epsilon sin\left(\frac{\pi x}{l}\right) \quad (1)$$

The question also states that:

$$y_o = 5sin\left(\frac{2\pi x}{3}\right) \quad (2)$$

Substituting (2) into (1) gives

$$5sin\left(\frac{2\pi x}{3}\right) = \epsilon sin\left(\frac{\pi x}{l}\right)$$

where:

$$\epsilon = 5mm$$

and:

$$\frac{2\pi x}{3} = \frac{\pi x}{l} \quad (3)$$

Re-arranging (3) gives:

$$l = \frac{3}{2}m$$

as given in the question.

$$I = \frac{bd^3}{12} = \frac{25 \times 25^3}{12} = 32{,}552.08mm^4$$

From page 17 of lecture notes:

$$y = \frac{\epsilon sin\left(\frac{\pi x}{l}\right)}{\left(1 - \frac{P}{P_c}\right)} = \frac{\epsilon P_c sin\left(\frac{\pi x}{l}\right)}{(P_c - P)} \tag{4}$$

and:

$$P_c = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 200{,}000 \times 32{,}552.08}{1500^2} = 28{,}557.88N$$

where E = 200GPa for Steel

$$M = Py$$

$$\therefore M_{max} = Py_{max} \tag{5}$$

Substituting (4) into (5) gives:

$$M_{max} = P\frac{\epsilon P_c sin\left(\frac{\pi x}{l}\right)}{(P_c - P)} = 15{,}000 \times \frac{5 \times 28{,}557.88 \times sin\left(\frac{1}{2}\pi\right)}{28{,}557.88 - 15{,}000} = 157{,}977.57Nmm$$

Axial and Bending Stresses together gives:

$$\sigma_{max}^{compressive} = \frac{P}{A} - \frac{M_{max} \times y_{x-section}}{I} = -\frac{-15{,}000}{25 \times 25} - \frac{157{,}977.57 \times 12.5}{3.26 \times 10^{-8}}$$

$$\boxed{\therefore \sigma_{max}^{compressive} = -84.66MPa}$$